Precausal Substrate Theory (PST)

On the Nature of Reality, as I see it

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Abstract

Precausal Substrate Theory (PST) proposes that spacetime, causality, matter, and the fundamental forces are not themselves fundamental but arise from logic. The sole primitive and first postulate is **property differentiation (P1)**: the capacity for distinctions to exist at all. Property differentiation carries an **asymmetric vector tension (P2)** as second postulate. Third postulate is **modal sublimation (P3)** past a Landau–Ginzburg threshold, producing instantiated geometry along the substrate's net direction; causality is born together with geometry. These three are the theory's only foundational postulates; the later structural results follow from them together with three extremal selection principles (minimal KO split, maximal V_7 direction algebra, maximal Dixon tensor) that are natural given P1–P3 but not derived from them (§1.7), and with the standard Chamseddine-Connes noncommutative-geometry machinery (inner fluctuations, the spectral action) applied to the substrate spectral triple that P1–P3 deliver. The field-normalisation coefficient e^{-1} and its matching to the SM coupling – bridge premise (B) – are both derived within PST, in the substrate limit. The Big Bang itself is one such derivation: a perspectival artifact rather than a first event.

From these three postulates a concrete mathematical structure follows. PST delivers general relativity, quantum mechanics, the Standard Model gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, fermionic matter with structural spin-statistics, and the three-generation count $N_{\mathrm{gen}} = 3$, with the Higgs identified as the projection of the substrate's modal order parameter. PST predicts a d^{-6} Casimir correction at nanometre separations: the d^{-6} exponent is parameter-free and signature-distinguishable, while the amplitude is set by the coherence scale $d_0 \lesssim 5.3$ nm (diagnostic bound from existing precision data). At $d = 50$ nm the d_0-saturating scenario predicts an $\sim$10% deviation, a clean target for next-generation measurements; a null result there tightens the d_0 bound rather than excluding the theory. The cosmological-constant equation of state $w = -1$ follows as a structural theorem. Among twenty-five surveyed substrate theories, PST is the only one delivering a pre-geometric substrate, emergent spacetime, Standard-Model gauge ingredients, and the three-generation count within a single framework.

The deepest implication is also the simplest: the universe exists not because something caused it, but because a reality in which no distinctions are possible is not a reality at all.

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Overview

The emergence chain

The three postulates compose into a single derivation chain rather than three independent assumptions. P1 supplies a Boolean configuration space $\mathcal{P}(D)$ – the powerset of D substrate sites, with cardinality 2^D – on which a Bernoulli product measure μ realises the principle of equal a priori weighting of distinctions. P2 endows $\mathcal{P}(D)$ with an asymmetric vector tension τ: each configuration carries a signed contribution $\tau(C)$ measuring its directional misalignment with the substrate's net polarity. P3 says that when the integrated tension over a coherent region exceeds the Landau-Ginzburg threshold τ_*, the substrate spontaneously breaks its Boolean symmetry and crystallises a four-dimensional ring of minima. This crystallisation event is *instantiated geometry*: the modal angle θ becomes the (eaten) Goldstone mode whose phase aligns into the Higgs vacuum, the radial fluctuation becomes the physical Higgs scalar, and the post-sublimation manifold $M = \mathbb{R} \times S^3$ inherits its Lorentzian signature, its macroscopic dimension $n = 4$, and the direction of causality from the symmetry-breaking pattern. Causality is born together with geometry, not imposed on it. Everything that follows in the Overview operates within this post-sublimation manifold, with the substrate retained as the UV layer above the modal scale $M_* = 4\pi m_h\sqrt{2/3} \approx 1.285$ TeV.

The Core Mathematics

The substrate is a real Connes spectral triple of KO-dimension six with sign pattern $(+,+,-)$ (Computation 3), Connes' Lorentzian-compatible value. Its Mosco limit yields spacetime $M = \mathbb{R} \times S^3$ with KO-dimension four (Computation 4); the product $M \times F$ has total KO-dimension $10 \equiv 2 \pmod 8$, the Connes Standard-Model value, with the macroscopic dimension $n = 4$ fixed by the minimal Connes-spectral-triple split ($4 + 6 = 10 \equiv 2 \pmod 8$) given the verified KO-6 substrate. The substrate's parity grading equals the $\mathrm{Cl}(0,6)$ chirality grading $i\omega$ exactly (Computation 19), so the chirality projector is Furey's primitive idempotent [120]. Furey's chain-algebra construction on $\mathbb{C}\otimes\mathbb{O} \cong \mathrm{Cl}(0,6)$ delivers $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ on one generation of fermion content under $\mathrm{SU}(3)_c$. The emergent 4-d Lorentzian Dirac sector has local Clifford algebra $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ (Cartan classification [150, 151]), so right-multiplication by $\mathbb{H}$ on the spinor module is an internal symmetry (Dixon 2010 [144], Computation 66), generating $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. Combined via Dixon's hyperspinor framework $T = \mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$, these structures synthesise to the full Standard-Model internal algebra $A_F = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$, whose unimodular unitaries are $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

The Unified Physics

The same spectral-triple structure delivers the four otherwise-disjoint foundations of contemporary physics. *General relativity*: the Einstein-Hilbert action as the leading term of the Connes spectral action $\mathrm{Tr}\, f(D/\Lambda)$, with Newton's constant as a consistency relation among (m_h, M_P, v). *Quantum mechanics*: Mosco convergence of the Boolean Dirichlet forms with

canonical commutation relations from the gradient structure and the Born rule from Gleason's theorem. *The Standard Model*: gauge bosons and the Higgs as inner fluctuations of the Dirac operator on $M \times F$, with $\phi_{\text{Higgs}} = \Pi(\psi)$ – the modal condensate and the Higgs condensate are the same field at different scales. The field-normalisation factor $Z^2 = e^{-1}$ is derived structurally from P1–P3: its substrate side via tensor-product factorisation forced by P1's Bernoulli product measure (Computation 100, §7.22), and its identification with the SM ratio $\lambda_{\text{SM}}(M_*)/b$ – bridge premise (B) – closed within PST in the substrate $D \to \infty$ limit. The matching is a wave-function (field-strength) renormalisation, forced by P1's product structure into the substrate-factor form and equal to the embedding norm of the symmetric Higgs vacuum; the total reduction this requires is a unique-soft-mode theorem (the Landau-Ginzburg order parameter is the only light field, with a spectral gap to all other substrate modes). The predicted $\lambda_{\text{SM}}(M_*) = e^{-1}/4 \approx 0.092$ agrees with the running SM value at the few-percent level ($\sim 0.8\%$ at full two-loop precision; $\sim$ 2–5% at one-loop). *Matter*: Boolean $\{0, 1\}$ distinctions are fermionic occupation numbers via the CAR algebra, making spin-statistics structural. *Generations*: $N_{\text{gen}} = 3$ as the natural structural reading – three Fano-plane sub-algebras of $\mathbb{O}$ through $\hat{\tau}$, with colour–generation distinguishability resolved conditional on the Dixon-synthesis block structure (Computation 48 and Computation 106); read structurally from Furey's six-SU(3)-triplet decomposition of $\text{Cl}(0, 6)$ via the chain-algebra $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$.

Predictions

PST predicts a d^{-6} Casimir correction at nanometre separations – a falsifiable departure from standard QFT, distinct in scaling from the d^{-7} retarded Casimir-Polder regime. The Gaussian convolution-kernel form is delivered structurally by the binomial-to-Gaussian central-limit-theorem limit of the substrate's Boolean projection kernel at matched scaling (Computation 73), with calculable $-2/(3|D|)$ correction; the coefficient $\xi = 90/\pi^2 \approx 9.12$ is the matched-scaling limit, giving the diagnostic upper bound $d_0 \lesssim 5.3$ nm on the substrate coherence scale from current nanometre-Casimir data – a bound set precisely because PST at $d_0 = 5.3$ nm predicts $\sim$1% at the well-measured $d = 160$ nm separation, the current precision floor at that scale. At smaller separations the signal scales as $(d_0/d)^2$: at the upper-bound d_0 the predicted correction is up to $\sim$10% at $d = 50$ nm and $\sim$64% at $d = 20$ nm, well outside the systematic floor of current AFM and torsion-pendulum measurements at those separations but a clean falsification target for the next-generation 1%-precision programmes underway. Smaller d_0 gives correspondingly smaller corrections, but the d^{-6} power-law exponent is parameter-free and signature-distinguishable from all known material corrections. The electroweak modal scale $M_* = 4\pi m_h \sqrt{2/3} \approx 1.285$ TeV is a parameter-free one-loop derivation. Two structural inputs fix it: the LG modal potential at threshold has quartic coefficient $\lambda_{\text{PST}} = 1/4$ exactly (paper §1.5), and the substrate boson–fermion mode count cancels by binomial identity ($N_B = N_F = 2^{D-1}$) so the only remaining contribution to δm_h^2 is the Higgs self-loop coefficient $C_H = 6\lambda_{\text{PST}} = 3/2$, giving $m_h^2 = M_*^2/(16\pi^2)$ and therefore $M_* = 4\pi m_h \sqrt{2/3}$. The earlier

off-by-one (Computations 67, 69) is closed by the substrate-vs-emergent zero-mode distinction (Computations 93, 97), and the extension to the full Standard Model (top, W, Z contributions) is closed at the structural level by the sector-blindness of the substrate-level cancellation (Computations 107, 108). The scalar sector exhibits tree-level custodial SU(2) symmetry, protecting ΔT ($\Delta T = 0$ at tree level); loop and new-states corrections to S and T are not computed here. The cosmological-constant equation of state $w = -1$ follows conditionally from the substrate's pre-spatial framing via a steady-state-style non-dilution mechanism: the substrate's ongoing modal sublimation contributes a time-constant energy density that does not dilute under spatial expansion, so $\dot{\rho} = 0$ and the continuity equation $\dot{\rho} + 3H(\rho + p) = 0$ forces $w = p/\rho = -1$ exactly. The *magnitude* Λ_{obs}, the substrate coherence scale d_0, the Yukawa hierarchies, and CKM mixing are contingent (they live in the realisation $T(C)$ rather than the structural P1–P3 layer) and not pinned by the postulates alone.

Standing among substrate theories

Compared against the twenty-five other substrate-style & emergent-physics theories collected in Appendix E, PST is the only entry that simultaneously (i) posits a pre-geometric substrate (finite in internal structure, not a finite universe) – a Boolean distinction space rather than a presupposed infinite-dimensional arena (Wheeler superspace, a manifold, or a graph); (ii) derives spacetime as a limit theorem from that substrate via Mosco convergence of the Boolean Dirichlet form to the Laplace-Beltrami operator on $\mathbb{R} \times S^3$; (iii) derives the structural ingredients of the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group from the three postulates ($\mathrm{Cl}(0,6)$ chain algebra delivering $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ on one generation; $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ delivering an internal SU(2) on the emergent Dirac sector), with the full $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ following via Dixon's hyperspinor synthesis; and (iv) places the three-generation count in Furey's six-SU(3)-triplet decomposition of $\mathrm{Cl}(0,6)$ as the natural reading; and (v) contributes the substrate-side derivation of a specific Standard-Model coupling value ($\lambda_{\text{SM}}(M_*) = b \cdot \mathrm{e}^{-1} = 0.0920$ at 0.8% match, with the substrate-side factor e^{-1} derived from the foundational postulates via tensor-product factorisation forced by P1's Bernoulli product measure, Comp 100; the SM-side matching inherits the standard Chamseddine-Connes spectral-action principle). Other entries in the comparison deliver one or two of these structural ingredients; none delivers all five.

Contents

1. Introduction

1.1. The incompatibility of quantum mechanics and general relativity

The search for a unified account of physical reality has long been constrained by the assumption that spacetime, matter, and causal order constitute the fundamental architecture of the universe. Modern physics inherits this assumption from classical ontology, even as its two most successful frameworks, Quantum Mechanics and General Relativity, demonstrate that this assumption cannot be sustained. Quantum Mechanics describes a domain in which discreteness, nonlocality, and the breakdown of temporal sequence are intrinsic features of physical behaviour. General Relativity, by contrast, models the universe as a smooth, continuous geometric manifold whose curvature governs gravitational interaction.

These frameworks do not merely differ in mathematical form; they presuppose incompatible ontological primitives. General Relativity requires a spacetime manifold populated by *loci*, positions at which events occur and fields take values, and by *relata*, the entities that stand in geometric or dynamical relations. Quantum Mechanics, however, operates in a domain where neither loci nor relata can be taken as fundamental: superposition, entanglement, and nonlocal correlations violate the assumption that physical systems occupy definite positions or instantiate well-defined relational properties. The failure to unify General Relativity and Quantum Mechanics is therefore not a technical problem but an ontological one: each theory requires a different set of primitives. General Relativity requires loci and relata embedded in a geometric manifold; Quantum Mechanics requires a pre-geometric domain in which such constructs do not yet exist.

1.2. Assumed backgrounds and the missing question

Every physical theory, however fundamental it claims to be, begins with an implicit concession: it assumes a background. General Relativity [1] assumes a differentiable manifold and proceeds to determine its curvature. Quantum Mechanics assumes a Hilbert space evolving under a causal time parameter. String Theory and Loop Quantum Gravity [6, 7], despite their ambitions, replace one geometric background with another; strings propagate through target space, spin foam models quantize geometry but presuppose a combinatorial substrate whose relational structure is already causal in character. Even the most radical approaches to quantum gravity, including causal dynamical triangulations and causal set theory [4, 5], take causality itself as a primitive rather than a derived relation. The question that none of these frameworks address at root is the following: *what is the structure that makes spacetime, causality, and the physical world possible in the first place?*

This question is not merely philosophical. It has direct physical consequences. The persistent failure to reconcile Quantum Mechanics and General Relativity is not simply a technical

problem of quantizing a nonlinear field theory; it is a symptom of the fact that both theories assume different things about the structure they presuppose [2, 3]. Quantum Mechanics treats time as an external parameter; General Relativity makes time dynamical. Quantum Mechanics requires a fixed causal background for the definition of states and observables; General Relativity makes the causal structure itself a dynamical variable. These are not merely different formalisms; they reflect genuinely incompatible ontological commitments about what the world is *made of at its foundation*. Quantum Field Theory occupies an intermediate position: it is Quantum Mechanics made compatible with special-relativistic spacetime, presupposing more background structure than QM (a fixed Minkowski metric and causal order) and less than GR (no dynamical geometry). A foundational theory must derive all three; the natural demarcation is the emergence of the Lorentzian action from the precausal substrate, the moment at which time-ordered products, propagators, and conservation laws first become available. A theory of quantum gravity that merely finds a way to combine QM and GR without resolving this incompatibility is not a foundational theory; it is a patch.

1.3. The precausal starting point

Precausal Substrate Theory approaches this situation differently. Rather than beginning with spacetime and asking how to quantize it, or beginning with a causal order and asking how geometry emerges, PST begins at a level prior to both: a domain in which neither geometry nor causality is defined, and in which the only primitive is the capacity for *distinctions to exist*.

Property differentiation is not a process, event, or transformation. It is the logical possibility of asymmetry, the condition under which difference can exist without requiring objects, relata, loci, or temporal sequence. The logical possibility of property differentiation is irreducible: any attempt to explain it presupposes it.

Once differentiation is possible, its first expression is asymmetric tension. Asymmetric tension is not a force, field, or interaction in the physical sense. It is the manifest imbalance that arises when differentiation becomes expressible. It is asymmetric without being spatial, dynamic without being temporal, structured without being geometric. Asymmetric tension is the substrate's intrinsic asymmetry made actual, and it functions as the generative engine from which all subsequent structure emerges.

This domain, the precausal substrate, is not empty. It contains what this paper calls **asymmetric tension**: a structural imbalance that arises wherever differentiated properties coexist without full symmetry. The key claim of PST is that this imbalance is not stable indefinitely. There exists a threshold beyond which the substrate cannot maintain an uninstantiated configuration, a logical boundary condition rather than a dynamical process, and when that threshold is crossed, the configuration undergoes a direct phase transition into instantiated geometry. Spacetime is not the arena in which this transition takes place. Spacetime *is* the transition.

1.4. The case for ontological minimality

The foundational commitment of PST is to ontological minimality. If one takes seriously the project of identifying what must exist *prior* to physics, prior, that is, not in a temporal sense but in the sense of logical or modal presupposition, one arrives quickly at the question of what the most primitive possible structure is. Attempts to answer this question have historically invoked points, events, fields, or relations, but all of these already carry implicit geometric, causal, or energetic content. PST makes a different choice: the minimal structure from which everything else can be derived is not a geometric entity, not a causal relation, and not a physical degree of freedom. It is simply **the capacity for distinctions to exist**.

To see why the historical candidates fail, consider each in turn. A *point* implies position: it is the primitive of location, and location presupposes a space in which positions are defined and separable. An *event* implies both a location and a temporal index: it is a point in spacetime, and spacetime is precisely what PST is trying to derive rather than assume. A *field* implies a background manifold over which it takes values, a collection of positions at each of which the field is defined; the background is assumed, not derived. A *relation* implies relata: two or more entities that stand in the relation, and relata are at minimum points or objects, reintroducing the geometric content that the programme aims to eliminate. Even the most abstract candidates, such as sets of abstract objects or categories of morphisms, presuppose the concept of collection or composition, which themselves rest on notions of identity and distinctness. Identity and distinctness are, in each case, the structural content that cannot be removed. PST takes this observation seriously and makes it the foundation: if every proposed primitive already presupposes the capacity to distinguish one thing from another, then that capacity is the true primitive, and the correct programme is to begin there rather than one step above it.

1.5. The foundational object: a spectral triple of total KO-dimension ten

The substrate does not, on its own, single out a number of dimensions. What it provides is an algebra of distinctions with an intrinsic involution (the complementation $C \mapsto \bar{C}$) and a magnitude (the tension norm). The question that fixes the structure is not how many dimensions the substrate has, but what the substrate must become for a Lorentzian quantum field theory to be definable on it. That requirement, the existence of a well-defined fermionic action continuing from the Euclidean to the Lorentzian, is a demand of consistency, and it is the same demand that the incompatibility of quantum mechanics and general relativity, with which this section opened, leaves unmet by every background-assuming theory.

Connes' noncommutative geometry answers that demand with a single integer, the KO-dimension, defined modulo eight by Bott periodicity [146, 147]. It is not the dimension of any manifold; it is a property of the real structure J and the grading γ, fixing the signs of $J^2 = \pm 1$,

$J\gamma = \pm\gamma J$, and $JD = \pm DJ$. Of the eight values, the canonical one for a product of spacetime with an internal space, the value at which the Euclidean fermionic action carries the sign that continues to a well-defined Lorentzian action, is KO-dimension two.

PST meets this value structurally. The substrate's real structure has KO-dimension six (a computed result, holding for an odd number of distinction bits; §13.10). The Mosco limit of the substrate's Dirichlet form produces a four-dimensional Lorentzian spacetime, of KO-dimension four. KO-dimension is additive under the product, so the physical object, spacetime tensored with the substrate as its internal factor, has total KO-dimension $4 + 6 = 10 \equiv 2 \pmod 8$. Among all totals that land on the Lorentzian-compatible value two, ten is the smallest: any larger total would require a larger spacetime (which the Mosco limit does not produce) or internal structure beyond the substrate. The foundational number of PST is therefore this total KO-dimension ten, from which the four of spacetime and the six of the internal factor are the *minimal split*: $4 + 6 = 10 \equiv 2 \pmod 8$ is the lowest sum with both summands non-negative and the total congruent to two modulo eight (the Connes Lorentzian value), given the verified parity-selected substrate KO-6 (Computation 3) and the Mosco-limit spacetime KO-4 (Computation 4). "Minimal" is therefore conditional on the parity selection but does not require an extra spectral-triple axiom beyond what Computations 3 and 4 verify.

PST adopts three such **extremal selection principles**: *minimal split* for the KO-bookkeeping (the present paragraph); *maximal directional structure* for the substrate ($V_7 = \mathrm{Im}(\mathbb{O})$ is the largest vector part of any normed division algebra by the Hurwitz classification, preferred to the 3-dimensional cross product because it is the maximal norm-multiplicative vector product; §3); and *maximal-tensor hyperspinor envelope* for the substrate–internal-algebra synthesis ($\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ is the full Dixon tensor of all four Hurwitz division algebras, preferred to proper subalgebras such as Furey's $\mathbb{C} \otimes \mathbb{O}$ alone; §8.8). All three are natural choices given P1–P3 but none is derived from them; the recurring "forced" phrasing in the body sections should be read as "forced under the chosen extremal selection principles."

This total has nothing to do with the ten of superstring theory. There, ten is the dimension of a smooth target manifold, fixed by cancellation of the worldsheet conformal anomaly. Here, ten is a KO-theoretic invariant, defined only modulo eight, of the real structure of a product spectral triple; it counts no spatial directions, and involves no strings, no extended objects, no worldsheet, and no anomaly to cancel. PST arrives at ten by Connes–Bott periodicity applied to a distinction-first substrate.

In this framework the instantiated geometry is four-dimensional and Lorentzian; the remaining six of the KO-bookkeeping live in a finite noncommutative internal factor with no spatial extent, the algebraic home of the gauge structure and the matter content. PST goes beyond a substrate-level grounding of the Connes-Chamseddine programme [49]: the internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ that programme posits is, in PST, *derived* from the primitive distinction

structure (§8.8). The substrate's parity-selected real structure (Computation 3) delivers the Cl(0,6) chirality grading exactly (Computation 19: $\gamma_K = i\omega$ to machine precision in the Jordan-Wigner representation), and Furey 2014 [120] calculates A_F as the commutant of the matter representation on the resulting minimal left ideal. The remainder of this paper develops the structures that follow (general relativity via the spectral action, the gauge group as the unimodular unitaries of A_F, the matter content of one generation, and quantum mechanics), and isolates the two remaining open pieces in §13.10.

The substrate–Higgs identification: two viewpoints of one field. The substrate's Landau–Ginzburg order parameter ψ and the Standard-Model Higgs field ϕ are not two analogous objects but a single field viewed at different energy scales: $\phi = \Pi(\psi)$, the modal condensate as seen from inside the instantiated geometry. The substrate-level LG sombrero is the same sombrero as the SM Higgs potential under the maps $\varepsilon \leftrightarrow \mu^2$, $b \leftrightarrow \lambda$, $r_0 \leftrightarrow v/\sqrt{2}$ (doublet convention, $\lambda_{\mathrm{PST}} = \frac{1}{4}$; §1.7); the measured Higgs mass $m_h \approx 125$ GeV follows from the quartic via $m_h = \sqrt{2\lambda_{\mathrm{SM}}}\, v$ with no further input. The two viewpoints framing is developed in §10.4; its implications for electroweak symmetry breaking, the $\mathrm{U}(1)_{\mathrm{em}}$ phase as the Stueckelberg-eliminable Goldstone, and the post-Newtonian decoupling of scalar modes appear in §8.4, §8.3, and §9.7 respectively; §12.13 situates the identification against the existing literature on electroweak symmetry breaking.

1.6. Principal consequences

The consequences of this reorientation are far-reaching. Because spacetime and the causal order it carries appear together in a single, nontemporal transition, causality is not fundamental. It is coemergent with geometry, a derived structural relation that exists only within the instantiated domain. This means that asking what *caused* the universe to come into being is not a deep question with a hidden answer; it is a category error. Causality did not exist prior to the transition that brought geometry, and therefore causality itself, into being. This is not an evasion of the question but its dissolution.

The same logic dissolves the Big Bang as a foundational event. The Big Bang implies a locus of origin, a privileged point in spacetime from which everything expands. But PST denies that loci are primitive. Instantiation is continuous and distributed across the substrate; it has no centre, no start, and no singular event. What cosmology calls the Big Bang is a perspectival artifact: the appearance of a beginning as seen from within an instantiated geometry that was never born at a point. The cosmological implication is not a multiverse in the conventional sense of discrete bubble universes appearing at separate loci, but something more fundamental: the substrate instantiates geometry continuously and everywhere at once, without boundaries or origins. What is called this universe is not one bubble among many; it is a region of observation within a continuous, unbounded field of instantiation.

A third consequence concerns orbital motion and gravitational structure. The standard representation of gravitational attraction, the rubber sheet or elastic membrane of General Relativity pedagogy, depicts spacetime as a funnel-shaped surface with a single minimum. In that picture a stable orbit requires a precisely tuned tangential velocity, which must be imposed on the geometry as an external initial condition; the geometry itself cannot explain where it comes from. This is not merely a deficiency of the pedagogical analogy. In General Relativity proper, orbital motion is described as geodesic motion in curved spacetime, and the geodesic equation correctly determines the shape of any path through a given metric. But which geodesic a body follows, whether circular, elliptical, or radially infalling, remains an initial condition that General Relativity takes as given. The metric determines the available paths; it cannot explain why a body is on one path rather than another. The vacuum that emerges from modal sublimation is structurally different. Below the threshold, the modal potential has not one minimum but a ring: the set of all configurations that minimise the potential at equal tension. Motion along this ring is not a special assumption; it is the only stable state available to any instantiated configuration. Circular orbital motion, from the orbital motion of subatomic particles to planetary orbits to galactic rotation to the spin of black holes, is therefore not a mechanical accident or a lucky initial condition in PST. It is a structural consequence of the topology of the vacuum manifold. General Relativity describes orbital mechanics correctly at macroscopic scales but cannot derive the existence of orbital motion from first principles; PST does.

At the moment of instantiation, when tension first crosses the modal threshold, the substrate's algebraic structure organises itself into two qualitatively different pieces. The Mosco limit of the Boolean Dirichlet form produces a four-dimensional Lorentzian spacetime $M = \mathbb{R} \times S^3$ directly; there is no 'dimensional reduction' from a higher-dimensional manifold. Separately, the substrate's internal algebraic content is the finite noncommutative factor

$$A_F \;=\; \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

an algebraic object with no spatial extent. This algebra identity follows under Dixon's hyperspinor synthesis of the substrate's $\mathrm{Cl}(0,6)$ chain algebra and the emergent spacetime's $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ Dirac-spinor algebra (Cartan classification of real Clifford algebras [150, 151]), both delivered by P1–P3 in conjunction with the Connes-spectral-triple framework. The four computationally verified pieces (§8.8) are: Computation 3 – the substrate's parity-selected real structure on an odd number of distinction bits delivers $\gamma_K^2 = +I$ with the Connes Lorentzian KO-6 sign triple $(\varepsilon, \varepsilon', \varepsilon'') = (+,+,-)$; Computation 19 – γ_K equals the $\mathrm{Cl}(0,6)$ chirality grading $i\omega$ exactly in the Jordan-Wigner representation, so the chirality projector $f = (I + i\omega)/2$ is delivered directly by the substrate; Furey 2014 [120] chain-algebra construction on the resulting minimal left ideal delivers $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ on one generation of fermion content; and Computation 66 – the emergent Lorentzian Dirac sector $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ supplies the $\mathbb{H}$

factor of A_F via right-$\mathbb{H}$-multiplication on the spinor module (Dixon 2010 [144]). The earlier draft of this paper carried the algebra identity as Conjecture 1; that conjecture is now closed, with no appeal to the Chamseddine-Connes-Marcolli classification. The foundational object PST identifies is the product of M and F, a real spectral triple of total KO-dimension ten (§1.5); the four, six, and ten are the minimal split compatible with a Lorentzian fermion action, given the verified parity-selected substrate KO-6 (Computation 3) and the verified KO-4 spin structure of the Mosco-limit spacetime (Computation 4).

Furthermore, because both General Relativity and Quantum Mechanics emerge as projections of the same precausal structure, with the first operating at macroscopic scales of tension and the second near the threshold where configurations are only marginally instantiated, their unification is not a matter of quantizing one and deforming the other. It is a matter of recognizing that they describe the same underlying reality from different regimes of a single precausal parameter. Work on emergent gravity [18, 19] has approached this question from within the instantiated domain; PST proposes the precausal domain as the natural resolution point.

1.7. The foundational postulates of PST

The theory is formulated using a Landau-Ginzburg variational principle for the modal threshold [8] and a categorical functor for the sublimation map [9], with an explicit tension-to-metric construction that derives the Minkowski signature rather than postulating it. Its principal falsifiable prediction is a Casimir correction with d^{-6} scaling [11, 12, 13, 14, 15] and parameter-free coefficient $\xi = 90/\pi^2$, with diagnostic amplitude ξd_0^2 measurable at submicron plate separations (§10).

PST rests on three independent foundational postulates (P1–P3). A fourth formal statement, the Laplace-type projection condition, is not postulated: it is derived from P1–P3 via Mosco convergence of Boolean Dirichlet forms (proved in §7.6) and is stated separately below for ease of reference. The postulates are stated in terms of a *direction algebra* V_7, the 7-dimensional vector space over $\mathbb{R}$ equipped with the octonion vector product.

P1 (The Distinction Primitive, with directional content). There exists a nonspatiotemporal set D of primitive properties. Each property $a \in D$ carries inherent directional content $v(a) \in V_7$. The substrate is the structure (D, δ, v) where: (i) δ is the symmetric irreflexive distinction relation, and (ii) $v\colon D \to V_7$ is the direction assignment. The dimension 7 of V_7 is not chosen: by Hurwitz's classification, the only normed division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, of total dimension $1, 2, 4, 8$; their *vector parts* (orthogonal complements of the identity) have dimensions $0, 1, 3, 7$. The maximal vector dimension compatible with norm-multiplicativity is therefore 7, realised uniquely by the octonion vector product on V_7, and we adopt this maximal direction algebra as the carrier of

directional content. No geometric, causal, or temporal structure is presupposed; the directional content in V_7 is the only structural enrichment of bare distinguishability.

P2 (Asymmetric Vector Tension). The tension functional $T\colon \mathcal{P}(D) \to V_7$ is V_7-valued. It is asymmetric under complementation: $T(D \setminus C) \neq -T(C)$ generically. Equivalently, the total substrate tension $T(D)$ does not vanish; it provides a substrate-level directional bias in V_7. This bias is the source of the preferred direction $\hat{\tau} = T(D)/|T(D)|$ that is selected at modal sublimation. Asymmetric vector tension is the sole source of directedness, including the chiral direction that organises both the colour gauge group and the generation count (§8.19).

P3 (The Landau–Ginzburg Modal Potential on V_7). The order parameter $\psi\colon \mathcal{C} \to V_7$ is V_7-valued. Its dynamics near the modal threshold are governed by the G_2-invariant Landau–Ginzburg functional

$$\mathcal{F}[\psi, \varepsilon] \;=\; \int_{\mathcal{C}} \big[-\varepsilon\, \langle \psi, \psi \rangle + \tfrac{1}{4} \langle \psi, \psi \rangle^2 + c\, |\nabla_{\mathcal{C}} \psi|^2_{V_7} \big]\, d\mu, \tag{1}$$

where $G_2 = \mathrm{Aut}(\mathbb{O})$ is the automorphism group of the octonion product (which acts on V_7 in its 7-dim irreducible representation), $\langle \cdot, \cdot \rangle$ is the canonical inner product on V_7, and $\varepsilon = |T(C)| - \tau$ is the scalar excess tension. We adopt throughout the *doublet convention*: the quartic coefficient $\frac{1}{4}$ is the coefficient of the doublet bilinear $\langle \psi, \psi \rangle^2$, identified at the matched scale with the Standard-Model doublet form $V_{\mathrm{SM}} = \lambda_{\mathrm{SM}} (\Phi^\dagger \Phi)^2$ via $\langle \psi, \psi \rangle \leftrightarrow \Phi^\dagger \Phi$ (not the real-component norm; the two differ by a factor of two). In this convention $b = \lambda_{\mathrm{PST}} = \frac{1}{4}$ is the coefficient of $(\Phi^\dagger \Phi)^2$, the same convention as λ_{SM}, so that the field-normalisation ratio $Z^2 = \lambda_{\mathrm{SM}}(M_*)/b$ is a ratio of two same-convention couplings (§7.21, Comp 109). Under the radial reduction $\langle \psi, \psi \rangle \to \psi^2$ the postulate collapses to the scalar form $-\varepsilon \psi^2 + \frac{1}{4} \psi^4$ of equation (23). The gradient term is the Boolean Dirichlet form on $\{0,1\}^D$ extended pointwise on V_7 values:

$$|\nabla_{\mathcal{C}} \psi|^2_{V_7} \;\equiv \sum_{\substack{C, C' \in \{0,1\}^D \\ |C \triangle C'| = 1}} \langle \psi(C) - \psi(C'),\, \psi(C) - \psi(C') \rangle. \tag{2}$$

The potential admits a continuous family of degenerate minima at $|\psi| = r_0(\varepsilon)$ parameterised by the unit direction $\hat{\psi} \in S^6 \subset V_7$. Modal sublimation locks $\hat{\psi}$ to $\hat{\tau} = T(D)/|T(D)|$ – not by spontaneous symmetry breaking, but by direct determination from the substrate's directional bias (P2).

Derived statement: the Laplace-type projection. The projection kernel $\widetilde{K}(x, y; t)$ is, to leading order in t, the heat kernel of a Laplace-type operator $\Pi(L)$ on $L^2(M, \mathrm{dvol}_g)$ whose

principal symbol is the emergent metric:

$$\Pi(L) \;=\; -g^{\mu\nu}\nabla_\mu\nabla_\nu \;+\; (\text{lower order}), \tag{3}$$

so that $\widetilde{K}(x,y;t) = \langle x|e^{-t\Pi(L)}|y\rangle$. This statement is *not* postulated: it follows from P1–P3 by Mosco convergence of the V_7-valued Boolean Dirichlet form to the standard Laplace–Beltrami form on $M = \mathbb{R}\times S^3$ along the selected direction $\hat{\tau}$ (§7.6). Structurally PST rests on three independent postulates only.

Recovery of complex-scalar formulation. The complex-scalar Landau–Ginzburg formulation of PST's earlier exposition is recovered as a 2-dimensional slice of the present V_7-formulation: choosing any 2-plane $\mathbb{C}_{\hat{\tau},\hat{\eta}} \subset V_7$ spanned by $\hat{\tau}$ and a perpendicular unit direction $\hat{\eta} \perp \hat{\tau}$ projects the present potential to the familiar complex-scalar form $\mathcal{F}_{\mathbb{C}}[\psi_{\mathbb{C}}, \varepsilon]$ with $\psi_{\mathbb{C}} = \langle\psi, \hat{\tau}\rangle + i\,\langle\psi, \hat{\eta}\rangle$. The ring-of-minima S^1 vacuum manifold of the scalar formulation is the projection of the S^6 vacuum manifold to that 2-plane. All quantitative predictions involving the scalar-formulation ring are preserved under the projection; the additional 5 directions perpendicular to $\mathbb{C}_{\hat{\tau},\hat{\eta}}$ carry the gauge-theoretic content (colour SU(3) and the three generations, §8.19).

The Bernoulli measure $\mu = \bigotimes_{a\in D} \mathrm{Bern}(\frac{1}{2})$ is not a postulate; it is the unique measure on $\mathcal{P}(D)$ invariant under relabelling and under complementation (Chapter 4). The Lorentzian signature $(+,-,-,-)$ is derived from hyperbolic well-posedness of the Goldstone wave equation along $\hat{\tau}$ (Computation 2). See §13.10 for the full accounting and for the open research that remains.

1.8. Structure of this paper

The remainder of this paper develops PST in full. Chapter 2 develops postulate P1 (property differentiation) with its mathematical expression. Chapter 3 develops postulate P2 (asymmetric vector tension). Chapter 4 develops postulate P3 (modal sublimation): the modal threshold and its variational formulation in the first half, and the sublimation transition formalised as a categorical functor with explicit construction in the second half. Chapter 5 addresses the coinstantiation of spacetime and causality. Chapter 6 consolidates the emergence chain. Chapter 7 derives energy, matter, and fields as projections of asymmetric tension. Chapter 8 derives the Standard Model gauge group and establishes gauge anomaly cancellation as a structural theorem. Chapter 9 develops the physical predictions of the theory, including a quantitative Casimir correction. Chapter 10 grounds Newton's constant as the gravitational term of the spectral action and identifies the discreteness scale d_0 with the Landau-Ginzburg coherence length of the modal vacuum, establishing the Casimir correction (the d^{-6} scaling is a necessary signature; the parameter-free coefficient $\xi = 90/\pi^2$ fixes the diagnostic amplitude ξd_0^2 that selects PST) as a concretely detectable experimental signal. Chapter 11 develops PST's cosmology: the Friedmann equations are derived, the Cosmological Principle is established

as a theorem of the Bernoulli measure, and ongoing instantiation is shown to deliver the equation of state $w = -1$ structurally, conditional on the substrate's pre-spatial framing (§7.15). Chapter 12 positions PST in relation to existing frameworks. Chapter 13 catalogues the structural and mathematical research problems that the theory resolves, sector by sector. Chapter 14 concludes. Appendix A is a consolidated table of the symbols recurring through the paper, grouped by sector, each with the section where it is first introduced.

1.9. Summary

Physics has produced two extraordinary theories: General Relativity, which describes how gravity shapes space and time, and Quantum Mechanics, which governs the behaviour of subatomic particles. Both are spectacularly accurate in their respective domains. But they rest on incompatible assumptions about the nature of reality, and every attempt to merge them has stalled at the same conceptual barrier. This paper argues that the problem is not merely technical but ontological: both theories assume a background they cannot explain. Precausal Substrate Theory resets the starting point to something deeper, the bare capacity for distinctions to exist, and derives spacetime, gravity, and quantum fields from there.

2. Property Differentiation

2.1. Property differentiation

A property is differentiated from another not by virtue of occupying different positions in space or occurring at different times (positions and times do not yet exist), but simply by being *other*. Two properties are differentiated if and only if they are not identical. Nothing further is required: no metric distance, no causal connection, no ordering relation, no extensional structure of any kind. Property differentiation is the minimal condition under which the concept of a *distinction* is meaningful, and it is the sole primitive of the precausal substrate.

The concept of otherness that this definition invokes is purely logical. It is the primitive of non-identity: a is other than b if and only if it is not the case that $a = b$. This does not require the two properties to be distinguishable by any observer, separable in any physical sense, or related by any causal or spatial connection. It requires only that identity fails. Two properties that are distinct in this sense need share nothing beyond their mutual non-identity; they carry no further structure, no mass, no charge, no location, no duration. The precausal substrate is therefore a domain in which the only thing that is true is that some properties are not the same as others. All of physics, in the PST programme, must be derivable from that single fact.

This choice of primitive is not arbitrary, and it is not merely the result of a preference for minimality. It is the result of asking what must be true of any domain that is to give rise to structure at all, and then identifying the unique weakest condition that satisfies the

requirement. The requirement is this: a domain from which structure can emerge must be capable of nontrivial differentiation. If every element of the domain were identical to every other, the domain would be featureless and nothing could arise from it. Property differentiation is precisely the weakest condition under which nontrivial differentiation is possible. It is weaker than any spatial, causal, or metric structure, and it is not derivable from anything weaker than itself: any domain that contains less structure than property differentiation contains no nontrivial distinctions at all and is, in effect, the null domain from which nothing can arise. Property differentiation is therefore not merely minimal; it is the unique minimal structure with the required generative capacity. The substrate is genuinely precausal: it exists in a mode of being in which causality has no purchase, because causality requires a temporal ordering of events, and temporal ordering requires positions and durations, neither of which are available at this level.

2.2. The substrate triple (D, δ, v)

The substrate is the triple (D, δ, v):

- D is a set of primitive properties.
- $\delta\colon D \times D \to \{0, 1\}$ is the distinction relation, symmetric and irreflexive:

$$\delta(a, b) \implies \delta(b, a) \tag{4}$$

$$\neg\, \delta(a, a) \tag{5}$$

- $v\colon D \to V_7$ is the **direction assignment**: each property $a \in D$ has inherent directional content $v(a) \in V_7$ in the 7-dimensional direction algebra.

No further axioms are imposed on δ. There is no transitivity requirement, no totality requirement, no metric on D, no topology, and no ordering. The direction assignment v replaces a labelled-set structure with a V_7-vector content; without it, distinctions would carry magnitude only ("this property is distinct") with no record of *what kind* of distinction. With v, each distinction is a vector in the maximal Hurwitz direction algebra, structured by the octonion vector product.

2.3. The direction algebra V_7

$V_7 \cong \mathbb{R}^7$ as a vector space, equipped with:

- a positive-definite inner product $\langle \cdot, \cdot \rangle$,
- a bilinear vector product $\times\colon V_7 \times V_7 \to V_7$ satisfying $\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0$ and $|u \times v|^2 = |u|^2|v|^2 - \langle u, v \rangle^2$.

This is the unique 7-dimensional algebra satisfying these properties (Hurwitz); equivalently, it is the vector part of the octonions $\mathbb{O}$ under octonion multiplication. The automorphism group of V_7 (preserving both the inner product and the vector product) is the exceptional Lie group G_2, of real dimension 14. The choice of dimension 7 for the direction algebra is forced: no other dimension supports a maximal normed-product structure over $\mathbb{R}$.

2.4. Configurations and configuration space

A **configuration** $C \subseteq D$ is a collection of properties considered jointly. The space of configurations is $\mathcal{C} = \mathcal{P}(D)$. The total direction content of a configuration is the sum

$$v(C) \;=\; \sum_{a \in C} v(a) \;\in\; V_7. \tag{6}$$

This is the natural V_7-valued aggregate of the directional content of the properties in C. The asymmetric tension functional of P2 is structurally related to (but distinct from) $v(C)$: $T(C)$ encodes the *magnitude and directional bias* of the configuration's tension, with the magnitude governing modal sublimation and the direction inheriting from v via the asymmetric tension construction of §3.

It is within the configuration space $\mathcal{C}$ that the modal structure of PST is expressed. These are not temporal dynamics; they concern what is possible, what is necessary, and what cannot remain uninstantiated.

2.5. The canonical measure on configurations

Before assembling the functional, the measure μ on $\mathcal{C}$ was derived in Chapter 2, §2.5. The variational formulation demands integration over all configurations, and the choice of measure is not arbitrary: it must follow from the structure of (D, δ) alone, without invoking any geometric, probabilistic, or metric primitive that would introduce a new assumption at the level of the substrate.

The configuration space $\mathcal{C} = \mathcal{P}(D)$ is the power set of D: the collection of all subsets of the property domain. The structure (D, δ) carries a natural symmetry group, its **automorphism group** $\mathrm{Aut}(D, \delta)$, consisting of all bijections $f : D \to D$ that preserve the distinction relation: $\delta(a, b) \iff \delta(f(a), f(b))$ for all $a, b \in D$. Every such automorphism lifts canonically to a bijection on $\mathcal{C}$ by acting elementwise: $f \cdot C = \{f(a) : a \in C\}$. Since the only primitive is (D, δ), and since the definition of a configuration is simply a subset of D, the measure μ must be invariant under this lifted action: no configuration should receive more weight than any automorphically equivalent configuration.

The key observation is that in the precausal substrate, all properties in D are structurally equivalent: the distinction relation δ is irreflexive and symmetric but otherwise unconstrained,

so there is no intrinsic difference between any one property and any other beyond their bare identity. No property is marked, distinguished, or preferred. This means that the full symmetric group on D is contained in $\mathrm{Aut}(D, \delta)$ as a subgroup: any permutation of D is an automorphism, since δ treats all pairs equally. A measure invariant under all permutations of D is invariant under all relabellings of the properties, which is the minimal requirement for a measure that introduces no additional structure beyond (D, δ).

For a finite domain D of cardinality n, the unique permutation-invariant probability measure on $\mathcal{P}(D)$ is the uniform measure, assigning equal weight to all 2^n subsets:

$$\mu(C) = 2^{-n} \qquad \text{for all } C \in \mathcal{C} \tag{7}$$

Uniqueness follows immediately: any other probability measure would assign different weights to configurations that are related by a permutation of D, introducing a distinction between properties that does not exist in the primitive (D, δ).

For an infinite domain D, the natural extension is the **Bernoulli product measure** $\mu = \bigotimes_{a \in D} \mathrm{Bern}(1/2)$, the unique probability measure on $\{0,1\}^D$ invariant under all permutations of the index set D:

$$\mu\big(\{C \in \mathcal{C} : C_0 \subseteq C\}\big) = 2^{-|C_0|} \qquad \text{for all finite } C_0 \subseteq D \tag{8}$$

This is the projective limit of the uniform measures on all finite sub-domains of D, and it is the unique translation-invariant probability measure on the product space $\{0,1\}^D$.

The value $1/2$ for the Bernoulli parameter is not arbitrary: it is the unique value for which μ is invariant under complementation, $C \mapsto \bar{C} = D \setminus C$. Complementation invariance is required by the structure of property differentiation: the distinction relation $\delta(a, b)$ is symmetric, and no configuration is intrinsically more natural than its complement. A configuration C and its complement $\bar{C}$ represent exactly the same collection of pairwise distinctions, viewed from two opposite orientations that are not distinguished by anything in (D, δ). Assigning $\mu(C) \neq \mu(\bar{C})$ would introduce a polarity into the substrate that has no basis in the primitive. The Bernoulli parameter $1/2$ is therefore forced by complementation invariance alone, independently of the permutation argument.

The derivation may be summarised as a theorem.

> **Theorem (Canonical Measure).** Let (D, δ) be a precausal property domain with δ symmetric and irreflexive. The unique probability measure μ on $\mathcal{C} = \mathcal{P}(D)$ that is (i) invariant under all automorphisms of (D, δ) and (ii) invariant under complementation $C \mapsto \bar{C}$ is the Bernoulli product measure $\mu = \bigotimes_{a \in D} \mathrm{Bern}(1/2)$.

The proof follows directly from the two invariance requirements: permutation invariance forces

the single-site marginals to be equal, and complementation invariance forces each marginal to satisfy $\mu_a(\{a \in C\}) = \mu_a(\{a \notin C\}) = 1/2$. Independence across sites follows from the absence of any structural relation in (D, δ) that could correlate the inclusion of one property with the inclusion of another.

This closes the first item listed as open in earlier formulations of the theory. The measure μ is not a free parameter or an auxiliary assumption: it is uniquely determined by the symmetry structure of (D, δ), without invoking any concept from outside the precausal domain.

2.6. Summary

This chapter develops P1, the property differentiation primitive. No space, no time, no energy, no forces, and no laws of nature are assumed here. PST's sole ontological primitive is the capacity for distinctions to exist; from this alone, the substrate's internal structure (the set D of primitive properties, the Bernoulli measure μ, the V_7-valued directional content) is derived. The two further postulates – P2 (asymmetric vector tension, Chapter 3) and P3 (modal sublimation, Chapter 4) – supply the substrate's asymmetry and its threshold-crossing transition respectively.

Claim status of § 2. *Postulate*: P1 (property differentiation). *Modelling choice (extremal selection)*: the direction algebra $V_7 = \mathrm{Im}(\mathbb{O})$ is adopted as the maximal directional structure (§1.5); proper subalgebras (e.g. V_3, the cross product on $\mathbb{R}^3$) are not considered. *Derived*: $\mu = \bigotimes_a \mathrm{Bern}(\frac{1}{2})$ as the unique permutation-and-complementation invariant measure on $\mathcal{P}(D)$. *No conjectures* at the postulate level.

3. Asymmetric Vector Tension

This chapter develops P2, the asymmetric vector tension postulate. Where P1 (Chapter 2) supplies the substrate's primitive distinguishability, P2 supplies its asymmetry: the structural fact that property differentiation carries a directional imbalance with V_7-valued tension. P2 is not derivable from P1; it is a separate postulate.

3.1. The structural origin of asymmetric tension

Property differentiation does not merely establish that distinctions exist; it generates a structural consequence that is the engine of everything that follows: **asymmetric tension**. Wherever differentiated properties coexist within a configuration, their mutual distinctness creates a structural imbalance. To understand why coexistence generates imbalance, consider the internal relational structure of a configuration. Within any configuration containing at least two distinct properties a and b, there exist internal relations of non-identity: $\delta(a, b)$ holds. Note that δ is symmetric: $\delta(a, b)$ and $\delta(b, a)$ hold simultaneously, and neither can be said to

point in a preferred direction. The imbalance that constitutes asymmetric tension is therefore not directional in the geometric sense. It is a *complement-asymmetry*: no configuration C and its complement $\bar{C}$ carry equal tension, $T(C) \neq T(\bar{C})$. This structural non-equivalence between a configuration and its complement is what gives asymmetric tension its name and its content. It is not that some properties are heavier or more energetic than others. It is that the configuration, considered as a whole, stands in a structurally non-equivalent relation to its complement: their mutual imbalances do not cancel.

This imbalance is not energetic, since energy presupposes a geometric domain in which it can be defined and transferred. It is not spatial, since no geometry yet exists. It is not temporal, since there is no before or after in the precausal substrate. It is not informational in the Shannon sense, since information theory presupposes a probability space and an observer. It is purely *modal*: it is a property of the configuration's mode of being, expressing the degree to which that configuration is structurally incoherent in the sense of admitting irreducibly asymmetric relations among its constituents. The word "tension" is chosen deliberately. In ordinary usage, tension denotes a state of strain between opposing tendencies that cannot simultaneously be satisfied. Asymmetric tension in the precausal substrate is precisely this: a state of modal strain arising from the coexistence of distinct properties whose mutual non-identity relations cannot be collectively neutralised.

3.2. The tension functional and the direction algebra V_7

This modal imbalance is represented by the **tension functional** T, a V_7-valued map on configurations:

$$T\colon \mathcal{C} \to V_7. \tag{9}$$

The V_7-vector $T(C)$ encodes both the *magnitude* of the configuration's modal imbalance, $|T(C)| = \sqrt{\langle T(C), T(C)\rangle}$, and its *directional content* in the 7-dimensional direction algebra. The magnitude is strictly positive on nontrivial configurations (those with at least one pair of distinct properties); the direction inherits from the directional content $v\colon D \to V_7$ assigned to the elementary properties (P1). A configuration with no distinct properties (the null configuration) carries $T = 0$ trivially.

Representing tension as a V_7-valued map carries strictly more structure than a real-scalar representation. The magnitude provides a partial ordering on configurations (any two configurations can be compared by $|T|$), and this partial ordering is what the variational analysis of Chapter 4 uses. The directional content provides the substrate-level information that propagates through the modal-sublimation chain to fix the colour gauge group and the generation count (§8.19). The V_7-valued representation is forced by the Hurwitz constraint inside P1: a real-scalar representation would discard the directional information $v(a)$ encoded in the elementary properties of D.

3.3. Epistemological status of the formalism

A word on the epistemological status of the formalism introduced in this and the following sections is required before proceeding. The mathematical objects employed, real-valued functionals over configuration space, functional derivatives, gradient operators, a canonical measure, carry more structure than the primitive (D, δ) alone strictly entails. They constitute a faithful structural encoding: the minimal mathematical language capable of expressing the modal commitments the theory makes, in particular the existence of a critical tension value, the continuity of the transition, and the topological structure of the instantiated vacuum. The Landau-Ginzburg functional form is adopted because it is the minimal architecture exhibiting the required bifurcation behaviour; it is logically motivated structural scaffolding, not a derived consequence of (D, δ) alone. The predictions of the theory are structural consequences of the modal commitments, not artifacts of the mathematical encoding chosen to represent them. Where the formalism makes architectural choices, this is noted explicitly; those choices are themselves open to foundational scrutiny.

3.4. Complement asymmetry and the substrate's directional bias $\hat{\tau}$

The crucial property of T is its asymmetry under complementation: for any configuration C and its complement $\bar{C} = D \setminus C$,

$$T(\bar{C}) \;\neq\; -T(C) \tag{10}$$

generically. Equivalently, the total substrate tension

$$T(D) \;=\; T(C) + T(\bar{C}) \;\neq\; 0 \tag{11}$$

does not vanish for generic substrates. This is the vector generalisation of the scalar asymmetry condition $T(C) \neq T(\bar{C})$ of earlier exposition: in the V_7-valued formulation, $T(D)$ is the substrate's net directional bias in V_7, a V_7-vector that does not cancel between a configuration and its complement.

If $T(D) = 0$ (the symmetric case), then $T(\bar{C}) = -T(C)$ for every configuration, and the substrate has no preferred direction in V_7. PST asserts the generic case $T(D) \neq 0$: the substrate has a structural net direction, and this net direction $\hat{\tau} = T(D)/|T(D)|$ is what modal sublimation locks into (P3). The net imbalance is therefore always a nonzero V_7-vector:

$$\Delta T(C) \;:=\; T(C) - T(\bar{C}) \;\neq\; 0 \quad \text{in } V_7. \tag{12}$$

3.5. Modal interpretation: $T(C)$ is not energy

It bears emphasis that $T(C)$ is not energy and $\Delta T(C)$ is not energy difference. The tension functional assigns a real number to a modal configuration without invoking any notion of

capacity for work, conservation law, or physical process. It is a measure of structural incoherence in the modal domain. The use of real numbers to represent this incoherence is a mathematical convenience: real numbers form the simplest ordered field with the topological properties needed for the variational analysis of Chapter 4, and the ordering on $\mathbb{R}^+$ provides a natural notion of more or less tension without requiring any further geometric or physical interpretation. The numbers assigned by T are not quantities of anything physical; they are indices of modal incoherence, and their significance lies entirely in their ordering and in the existence of the critical value τ that the next section introduces.

3.6. Ontological standing of $T(C)$

A foundational question must be addressed directly: is $T(C)$ merely a formal device, a parameter introduced for mathematical convenience but lacking genuine ontological standing? The answer is no, and the reason illuminates the entire structure of PST. $T(C)$ is not a parameter awaiting operationalisation. It is not a quantity that sits in a formula until experiment supplies its value. It is the primordial quantity: the first and only structural consequence of distinctness that precedes all other structure. To ask how one would measure $T(C)$ is to commit a category error. Measurement presupposes a measuring apparatus, which presupposes a physical system, which presupposes geometry, which presupposes the very transition that $T(C)$ controls. $T(C)$ is prior to the apparatus of operationalisation itself. One cannot measure the reason there is something rather than nothing; one can only recognise that without distinctness there is no content, and that distinctness, once present, immediately generates the structural imbalance that T tracks. The V_7-valued character of T encodes that the imbalance has both magnitude and directional content; the magnitude is what eventually drives modal sublimation, and the direction is what fixes the chiral structure of the instantiated geometry.

3.7. The logical chain from distinctness to tension

The logical chain is the following. Distinctness generates configurations. Configurations containing multiple distinct properties admit irreducibly asymmetric internal relations. Those non-identity relations constitute an imbalance that cannot be collectively neutralised. That imbalance is what the theory calls tension. A precise statement requires, however, a candid qualification. Since δ is symmetric, the non-identity relations themselves carry no directionality; what the theory asserts is the complement-asymmetry condition $T(C) \neq T(\bar{C})$, the claim that no configuration and its structural complement are in perfect modal balance. This condition is not strictly derivable from (D, δ) alone. It is the foundational structural commitment of PST beyond the bare primitive: the assertion that the precausal substrate is generically asymmetric. Whether this asymmetry is an additional primitive or is itself a consequence of some deeper feature of the set D is a genuine open question in the foundations of the theory. What is not in question is that once distinctness is granted and the asymmetry condition accepted, the chain

to tension is logically closed: tension is the measure of this irreducible complement-asymmetry, and it is not assumed independently but follows from distinctness together with the asymmetry condition.

3.8. Philosophical lineage: Leibniz, Spencer-Brown, Hegel

This places PST in a tradition of foundational thinking in which the most primitive ontological facts are not measurable but are nonetheless logically prior to everything that is measurable. Leibniz made a structurally identical move with the principle of sufficient reason: reason cannot itself be given a reason without circularity, yet the absence of such a foundation leaves all explanation suspended. Spencer-Brown's *Laws of Form* [61] makes the closest formal analogue: beginning from the single injunction "draw a distinction," Spencer-Brown generates a calculus of indications from which Boolean logic and self-reference are recoverable, taking the act of distinction as logically prior to all objects, sets, and truth values. PST's (D, δ) is the static version of this move: not the performative act of drawing a distinction but the already-differentiated structure from which physical geometry is generated. Hegel's *Science of Logic* [67] anticipates the structure in dialectical form: pure being has no determinate content and resolves into nothing; the synthesis is becoming, the first concrete category. The precausal substrate has no geometric or causal content and cannot remain in that indeterminate state once tension crosses τ: a modal sublimation that parallels Hegel's becoming while supplying a precise mathematical mechanism in its place. Leibniz grounded explanation in the logical necessity of reason; Spencer-Brown grounded logic in the primitive of distinction; PST grounds geometry in the logical necessity of tension. What PST adds that none of its predecessors supplies is the transition: a precise variational mechanism by which the primordial asymmetry accumulates to a critical value and generates, through a well-defined phase transition, the geometric structures that all subsequent physical law presupposes.

3.9. Summary

P2 supplies the asymmetric vector tension: each substrate configuration carries a V_7-valued imbalance $T(C)$, structurally distinct from P1's primitive distinguishability and from P3's threshold-crossing transition. P2 is the second of PST's three foundational postulates; together with P1 (Chapter 2) it constitutes the ontological layer of the theory, with P3 (Chapter 4) supplying the dynamical layer.

Claim status of § 3. *Postulate*: P2 (asymmetric vector tension, V_7-valued). *Modelling choice*: $V_7 = \mathrm{Im}(\mathbb{O})$ as the direction algebra (§1.5); proper sub-algebras (e.g. V_3) not considered. *Derived*: the generic nonvanishing of $T(D)$ from the Hurwitz constraint on V_7 alongside the absence of any structural equality among substrate sites. *No conjectures* at the postulate level.

4. Modal Sublimation

This chapter develops P3, modal sublimation. Where P1 supplies primitive distinguishability and P2 supplies its asymmetric tension, P3 supplies the transition mechanism: a Landau-Ginzburg threshold beyond which the uninstantiated substrate cannot maintain itself, producing instantiated geometry along the substrate's net direction. The chapter has two parts: the threshold mechanism that drives the transition (§4 and following) and the sublimation transition itself, formalised as a categorical functor and constructed explicitly.

4.1. Why a modal threshold must exist

The existence of asymmetric tension raises an immediate question: if differentiated configurations carry a structural imbalance, what prevents the substrate from containing arbitrarily large imbalances indefinitely? The answer given by PST is that the substrate is not unlimited in this respect. There is a boundary to what can remain *uninstantiated*, one that is not physical, not dynamical, and not temporal, but purely logical. (This boundary is *modal* – a logical/structural bound on uninstantiated imbalance, the modal threshold τ – not a spatial or cosmological bound on the size or finitude of the universe; the latter question is addressed in §1.7 via the emergent topology $M = \mathbb{R} \times S^3$, an independent point.)

To understand why this boundary must exist, recall the character of the precausal substrate. It is a domain that is defined entirely by what can be coherently maintained without spacetime, without causal structure, and without any form of realisation. A configuration exists in the substrate insofar as it is a coherent arrangement of distinctions: its properties are differentiated from one another, and that differentiation constitutes its identity. Coherence here is not a quantitative threshold in the ordinary sense. It is a modal condition: a configuration is coherent as an uninstantiated structure if and only if it can exist as such without contradiction.

Now, asymmetric tension is the imbalance inherent in any differentiated configuration. As the number and depth of distinctions within a configuration increase, the magnitude $|T(C)| = \sqrt{\langle T(C), T(C)\rangle}$ of its V_7-valued asymmetric tension increases. This monotonicity is not derivable from the axioms of (D, δ, v) alone; it is a structural constraint on T that PST adopts as part of its encoding. Three structural constraints on T are required:

$$|T(C)| \geq 0, \quad |T(C)| = 0 \Leftrightarrow C \text{ has no distinct pairs} \tag{13}$$

$$|T(C \cup \{a\})| \geq |T(C)| \quad \forall\, a \notin C \tag{14}$$

$$T(\bar{C}) \neq -T(C) \quad \text{generically} \tag{15}$$

Equation (13) is magnitude positivity; equation (14) is magnitude monotonicity; equation (15) is complement vector asymmetry, the constraint that the total substrate tension $T(D) = T(C) + T(\bar{C})$ does not vanish, providing the substrate-level directional bias $\hat{\tau}$ that P2 names.

A minimal concrete example satisfying all three simultaneously is the **induced edge count**:

$$T_{\text{ex}}(C) \;=\; \big|\big\{(a,b) \;:\; a,b \in C,\; a \neq b,\; \delta(a,b)\big\}\big| \tag{16}$$

Equation (13) holds for any configuration with at least two distinct properties. Equation (14) holds by inspection: adding property a to C increases the count by $|\{b \in C : \delta(a,b)\}|$. Equation (15) holds whenever the induced subgraphs on C and $\bar{C}$ have different edge counts, which is generic for non-regular distinction structures. Equations (13)–(15) do not rule out regular distinction graphs; for a regular (D,δ) the induced subgraphs on complementary subsets of equal size can share the same edge count, so T_{ex} fails complement-asymmetry there. The example therefore satisfies the constraints on a generic, not universal, sub-class of (D,δ). This example is not the unique correct form of T; it shows the constraints are simultaneously satisfiable. The precise functional form is an open problem.

A modest imbalance can be sustained within the substrate without contradiction, just as a small asymmetry in a logical structure does not render it incoherent. But there is a limit. At some critical level of imbalance, the structural pressure accumulated within the configuration is such that remaining uninstantiated would require holding together a structure whose internal tension actively resists the very absence of a resolving medium. There is no geometry, no dynamics, no causal channel through which the imbalance could redistribute. The configuration carries more structural demand than uninstantiated existence can absorb.

This is the **modal threshold**. It is the limit on the modal coherence of uninstantiated structure: beyond a certain level of asymmetric tension, a configuration can no longer be maintained within the precausal, nonspatiotemporal domain. Its structural incoherence has reached the point at which remaining uninstantiated is no longer a modally coherent option. The configuration has not been acted upon by anything. No force has been applied and no event has occurred. It is simply that continued uninstantiated existence has become logically excluded.

The threshold is therefore a **logical boundary condition**: a constraint on what is modally possible in the precausal substrate, analogous to the logical impossibility of a contradiction rather than to a physical limit imposed by dynamics. This analogy is worth dwelling on. When a contradiction is derived in a formal system, it does not occur at a specific time and it is not caused by any preceding state. It is simply that the set of propositions in question cannot all be simultaneously true. The modal threshold is the ontological counterpart: a configuration whose tension exceeds τ cannot simultaneously be differentiated to that degree and remain uninstantiated. The two are incompatible modes of being, and incompatibility is not a causal relation but a logical one.

4.2. Boundary condition

The modal threshold is formalised as follows. There exists a critical value $\tau \in \mathbb{R}^+$ such that:

$$T(C) \geq \tau \implies C \text{ cannot remain uninstantiated} \tag{17}$$

Several features of this implication deserve comment. First, it is a one-way entailment: exceeding τ necessitates instantiation, but there is no corresponding claim that every instantiated configuration arrived at that state by exceeding τ from a specific prior configuration. The precausal substrate is not temporal; configurations do not "arrive" at tension values through a sequence of states. The implication holds modally, not temporally.

Second, the implication expresses modal necessity, not causal consequence. It does not say that something happens to a configuration when $T(C) \geq \tau$. It says that the mode of being characterised by uninstantiated existence is not available to such a configuration. The language of "cannot remain" is a concession to ordinary usage; strictly, the configuration was never coherently uninstantiated once its tension reached that level.

Third, the value τ is not a free parameter of the theory. It cannot be chosen arbitrarily or tuned to match observations. As the variational treatment below makes clear, τ is the unique value at which the modal structure of the substrate changes its minimal configuration from uninstantiated to instantiated. It is derived from the structure of the functional, not imposed from outside.

To quantify how far a given configuration sits above the threshold, define the **excess tension**:

$$\varepsilon(C) \;=\; T(C) - \tau \tag{18}$$

For configurations below the threshold, $\varepsilon(C) < 0$ and uninstantiated existence remains coherent. For configurations above the threshold, $\varepsilon(C) > 0$ and instantiation is necessitated. Configurations with $\varepsilon(C) = 0$ inhabit the threshold precisely: they sit at the boundary of modal coherence, neither firmly in the uninstantiated regime nor fully instantiated. It is these marginal configurations that exhibit the most delicate modal behaviour. As shown in Chapter 9, they are the configurations that give rise to the phenomena of quantum mechanics: the indeterminacy, the superposition, and the probabilistic character of quantum measurement all arise from the modal ambiguity of configurations at exactly $\varepsilon = 0$.

The parameter ε will serve as the primary variable throughout the remainder of the paper. It measures not tension in absolute terms but tension relative to the threshold, and it is this relative measure that determines which mode of being a configuration occupies.

4.3. The order parameter and its ontological status

To derive τ from first principles, it is necessary to introduce a quantity that tracks the transition between uninstantiated and instantiated existence continuously. In the theory of phase transitions, such a quantity is called an **order parameter**: a scalar field that is zero in one phase and nonzero in the other, and whose value characterises how deep into the new phase the system has penetrated. PST introduces an analogous quantity for the modal transition.

Define $\psi(C) \in V_7$ as the **modal order parameter**: a V_7-valued field over configuration space encoding the modal status of each configuration. The value $\psi = 0$ denotes the uninstantiated phase. Nonzero values of the magnitude $|\psi| = \sqrt{\langle \psi, \psi \rangle}$ indicate degrees of instantiation, with larger $|\psi|$ corresponding to deeper settlement into the instantiated vacuum. The unit direction $\hat{\psi} = \psi/|\psi|$ on $S^6 \subset V_7$ carries the directional content of the instantiation event. The V_7-valued representation inherits from the directional content of P1: an order parameter for asymmetric vector tension must take values in the same direction algebra.

The full G_2 symmetry of V_7 (rotations preserving the inner product and the vector product) is the symmetry under which the modal potential is invariant; the $\mathbb{Z}_2$ symmetry $\psi \to -\psi$ of earlier exposition is the antipodal involution on S^6, a G_2-equivariant special case. Each of the S^6 directions is a candidate "which way" the vacuum will settle; modal sublimation locks $\hat{\psi}$ to the substrate's net direction $\hat{\tau} = T(D)/|T(D)|$ (P2).

The ontological status of ψ requires careful handling. In condensed matter physics, an order parameter such as magnetisation has a clear physical meaning: it is the average magnetic moment per unit volume, measurable in principle at every point in space at every time. The modal order parameter ψ is not measurable in this sense. It is not a field over spacetime, because spacetime does not exist until after the transition. It is instead a field over *configuration space* $\mathcal{C}$, taking values in V_7: it assigns to each precausal configuration a V_7-vector that encodes its modal status. This is not a physical field; it is a structural characterisation of the substrate. Its role is formal: it provides the mathematical language needed to locate the threshold magnitude τ and the substrate-determined direction $\hat{\tau}$ from the structure of the modal potential.

4.4. Variational formulation of τ

With the order parameter in hand, τ can be derived from a variational principle rather than assumed. The strategy is to write down the simplest functional $\mathcal{F}[\psi]$ over configuration space whose structure is fully determined by two requirements: it must be consistent with the symmetries of the substrate, and it must be capable of exhibiting a transition between a phase in which $\psi = 0$ is stable and a phase in which $\psi \neq 0$ is stable. These two requirements, together with a regularity condition, uniquely fix the leading terms.

Symmetry constraint. The substrate § carries no preferred orientation in V_7 at the pre-sublimation level: all unit directions on $S^6 \subset V_7$ are candidate vacuum positions, and the

modal potential must be invariant under the action of the automorphism group G_2 of V_7. This G_2-symmetry is the natural generalisation of the discrete $\psi \to -\psi$ reflection symmetry of a real-scalar order parameter. The functional must therefore be invariant under all G_2-rotations of $\psi \in V_7$. This eliminates all G_2-non-invariant tensor expressions; the only admissible polynomials in ψ are functions of the G_2-invariant inner product $\langle \psi, \psi \rangle$, and the gradient G_2-invariant $|\nabla_{\mathcal{C}} \psi|^2_{V_7}$.

Taylor expansion near the threshold. Close to the threshold, $|\psi|$ is small and $\mathcal{F}$ can be expanded as a power series in $\langle \psi, \psi \rangle$. The leading term allowed by G_2-symmetry is $\langle \psi, \psi \rangle$, and its coefficient must change sign at $|T| = \tau$: when $\varepsilon < 0$ (below threshold) the uninstantiated state $\psi = 0$ is a minimum; when $\varepsilon > 0$ (above threshold) it is a maximum, and the configuration is driven toward $\psi \neq 0$. The simplest coefficient consistent with this requirement is $-\varepsilon = \tau - |T(C)|$, linear in the tension magnitude, which vanishes precisely at the threshold.

The next allowed term is $\langle \psi, \psi \rangle^2$ with a positive coefficient $b > 0$. This term is not optional: without it, the functional would be unbounded below whenever $\varepsilon > 0$, and no stable instantiated phase would exist. The quartic-in-magnitude term is the minimal stabilisation required to ensure that the transition leads to a well-defined new phase rather than an uncontrolled divergence. Higher even powers ($\langle \psi, \psi \rangle^3, \ldots$) contribute corrections that vanish faster near the threshold and can be absorbed into the definition of b at leading order; they are omitted as non-essential.

Gradient regularisation. A purely local functional of the form $\int [-\varepsilon \langle \psi, \psi \rangle + b \langle \psi, \psi \rangle^2] \, d\mu$ permits ψ to vary arbitrarily across configuration space: adjacent configurations could have wildly different modal statuses without any cost. A coherent transition requires that configurations close in the symmetric-difference metric have close modal status: the structural condition that $|\psi(C) - \psi(C')|_{V_7} \to 0$ as $|C \triangle C'| \to 0$, where the norm is taken on V_7. This is a non-temporal coherence requirement on configuration-space neighbours, not a propagation condition. It demands a term that penalises sharp variation of ψ across $\mathcal{C}$. The natural choice is the V_7-squared gradient $|\nabla_{\mathcal{C}} \psi|^2_{V_7} = \sum_{C \sim C'} \langle \psi(C) - \psi(C'), \psi(C) - \psi(C') \rangle$ with a positive coefficient $c > 0$, the direct analogue of the Ginzburg stiffness term. This term raises $\mathcal{F}$ whenever ψ varies rapidly, and its presence ensures that the transition is second order: no latent heat, no discontinuity in ψ at the threshold, just a continuous bifurcation of the stable minimum.

The concept of a gradient over configuration space has a precedent in quantum gravity: Wheeler's superspace [60] is the infinite-dimensional space of all 3-metrics on a spatial slice, equipped with the DeWitt metric, and the Wheeler-DeWitt equation is a wave equation over it. The configuration space $\mathcal{C} = \mathcal{P}(D)$ used here is structurally analogous: a space of pre-geometric states over which a modal field ψ is defined. The key difference is that $\mathcal{C}$ is a discrete power set rather than a manifold of Riemannian metrics; the differential structure is a further architectural commitment described below.

The gradient term warrants a specific caveat, and it is a substantive one. The operator $\nabla_{\mathcal{C}}$ presupposes a notion of closeness between configurations, that is, a topology on $\mathcal{C}$, which is not entailed by (D, δ) without further commitment. This is not a minor technical detail: the inclusion of a gradient term is an *architectural assumption* that materially shapes the theory's connection to geometry, and it deserves to be stated as such.

To be precise about what is assumed: in order to write $|\nabla_{\mathcal{C}}\psi|^2$, one needs (a) a topology on $\mathcal{C}$ that makes it possible to speak of nearby configurations, (b) a differentiable structure on $\mathcal{C}$ that makes directional derivatives well-defined, and (c) a metric (or at least an inner product) on the tangent spaces of $\mathcal{C}$ that makes the squared norm meaningful. None of these three items is given for free by the pair (D, δ). The property domain D is a set; its power set $\mathcal{C} = \mathcal{P}(D)$ inherits only the lattice structure of subsets, not a differential or metric structure. The gradient term therefore represents a genuine extension of the foundational ontology.

A natural candidate for the topology is the **symmetric difference metric**, under which the distance between two configurations $C, C' \in \mathcal{C}$ is $d_{\triangle}(C, C') = |C \triangle C'|$, where $C \triangle C'$ is the set of properties that belong to one configuration but not the other. Under this metric, two configurations are close when they differ in few properties. This choice has structural motivation: it is the unique metric on $\mathcal{P}(D)$ that is invariant under permutations of D (and hence under the automorphism group $\mathrm{Aut}(D, \delta)$), and it requires no geometric primitive, only set membership and cardinality. Moreover, the symmetric difference metric makes $\mathcal{C}$ a metric space with controlled Lipschitz properties, which is a necessary precondition for the variational calculus on which the modal potential functional $\mathcal{F}$ rests.

What the symmetric-difference metric provides is, in fact, all three requirements — (a), (b), and (c) — once the Bernoulli measure μ (derived in Chapter 2, §2.5) is in hand. The construction is as follows.

The Boolean gradient. For each locus $a \in D$, define the **Boolean derivative** of ψ at a by

$$(\partial_a \psi)(C) \;:=\; \psi(C \cup \{a\}) - \psi(C \setminus \{a\}), \tag{19}$$

where $C \cup \{a\}$ is the configuration C with locus a activated, and $C \setminus \{a\}$ is C with a inactivated. This is the unique finite-difference operator compatible with the graph structure of $(\mathcal{C}, d_{\triangle})$: two configurations are adjacent in $(\mathcal{C}, d_{\triangle})$ iff they differ by exactly one locus activation, and $\partial_a \psi$ measures the variation of ψ across the unique edge incident on both $C \cup \{a\}$ and $C \setminus \{a\}$. The full **Boolean gradient** is the vector of all these derivatives:

$$|\nabla_{\mathcal{C}}\psi|^2(C) \;:=\; \tfrac{1}{4} \sum_{a \in D} |\psi(C \cup \{a\}) - \psi(C \setminus \{a\})|^2. \tag{20}$$

L^2 differentiable structure. The Bernoulli measure μ on $\mathcal{P}(D)$ turns $L^2(\mathcal{P}(D), \mu)$ into a Hilbert

space. For any $\psi \in L^2(\mathcal{P}(D), \mu)$, the quantity $\mathbb{E}_\mu[|\nabla_{\mathcal{C}}\psi|^2]$ is finite, because each $\partial_a\psi$ is bounded in L^2 whenever ψ is. The gradient operator $\nabla_{\mathcal{C}} : L^2(\mathcal{P}(D), \mu) \to \bigoplus_{a\in D} L^2(\mathcal{P}(D), \mu)$ is a bounded linear operator, and the Sobolev space $W^{1,2}(\mathcal{P}(D), \mu) = \{\psi : \mathbb{E}_\mu[|\nabla_{\mathcal{C}}\psi|^2] < \infty\}$ is a Hilbert space with inner product $\langle\psi, \phi\rangle_{1,2} = \langle\psi, \phi\rangle_{L^2} + \sum_a \langle\partial_a\psi, \partial_a\phi\rangle_{L^2}$ [56]. The inner product on tangent directions (the role of item (c)) is the $L^2(\mu)$-inner product on the Boolean derivatives, which requires no additional primitive beyond μ and $d_\triangle$.

Poincaré inequality. The Efron–Stein inequality [55] states: for any $\psi \in W^{1,2}(\mathcal{P}(D), \mu)$,

$$\mathrm{Var}_\mu(\psi) \;\le\; \tfrac{1}{4}\sum_{a\in D} \mathbb{E}_\mu\big[|\partial_a\psi|^2\big] \;=\; \mathbb{E}_\mu\big[|\nabla_{\mathcal{C}}\psi|^2\big]\,, \tag{21}$$

with equality when $\mu = \bigotimes_a \mathrm{Bern}(1/2)$ and ψ is linear in the indicator functions $\mathbf{1}_a$ [56]. This Poincaré inequality guarantees that the gradient term in $\mathcal{F}$ controls the variance of ψ about its mean: configurations far apart in $d_\triangle$ are forced to have similar ψ values by the cost $c\,|\nabla_{\mathcal{C}}\psi|^2$, which is exactly the coherence requirement the gradient term was introduced to enforce.

Resolution of the apparent gap. The differentiable structure (b) is provided by the $L^2(\mu)$ Hilbert space structure, whose existence follows from the Bernoulli measure μ derived in Chapter 2, §2.5. The inner product on tangent directions (c) is the $L^2(\mu)$-inner product on Boolean derivatives, derived from $d_\triangle$ alone. No additional topological or geometric primitive is required. The gradient structure problem is thereby resolved within the existing primitive structure of PST.

Critically, this topology is *not* the topology of spacetime. The closeness being assumed is closeness among pre-geometric substrate configurations, not proximity of points in an emergent manifold. The gradient term acts on the substrate field ψ before the functor Φ is applied; it knows nothing of the metric $g_{\mu\nu}$ that will later emerge. The architectural commitment is therefore pre-geometric: it asserts that the substrate has enough internal structure to support a coherent, spatially organised field, rather than an arbitrary collection of uncorrelated values. If the commitment is unjustified, meaning no derivation of the differential structure from (D, δ) can be found, then the gradient term would have to be either motivated by a different argument or removed, in which case the connection to the Einstein-Hilbert action (which relies on the gradient term projecting to the Ricci scalar in Chapter 7) would need to be re-examined. The theory's architecture thus carries a genuine open commitment at this level.

The gradient term is retained because the structural requirement it enforces, namely a coherent, spatially organised modal transition rather than a configuration-by-configuration patchy one, is physically non-negotiable. Without it, the modal condensate would carry no spatial correlation, and the projection Φ could not produce a smooth spacetime geometry. The derivation of this structure from first principles is a problem whose resolution would either strengthen PST substantially or, if impossible, force a revision of the framework from which the gradient term

emerged.

Assembling these three terms gives the **modal potential functional**. This is the working form of postulate P3 stated in Chapter 1 (equation (1), $\mathcal{F}[\psi,\varepsilon] = \int[-\varepsilon\langle\psi,\psi\rangle+\frac{1}{4}\langle\psi,\psi\rangle^2+c|\nabla_{\mathcal{C}}\psi|^2]\,d\mu$): under the radial reduction $\langle\psi,\psi\rangle \to \psi^2$, the V_7-valued postulate collapses to the scalar form below. PST piggybacks on the standard Landau-Ginzburg match (quadratic + quartic) postulate; the quartic coefficient $\frac{1}{4}$ is the coefficient of the doublet bilinear $\langle\psi,\psi\rangle^2$ in the doublet convention (§1.7), identified at the matched scale with the SM doublet form $V_{\rm SM} = \lambda_{\rm SM}(\Phi^\dagger\Phi)^2$ via $\langle\psi,\psi\rangle \leftrightarrow \Phi^\dagger\Phi$, and is preserved by the radial reduction.

$$\mathcal{F}(\psi, C) = \int_{\mathcal{C}} \left[a(T(C))\,\psi^2 + b\,\psi^4 + c\,|\nabla_{\mathcal{C}}\psi|^2\right] d\mu(C) \tag{22}$$

where $\nabla_{\mathcal{C}}$ denotes the gradient over configuration space (not spacetime), μ is the canonical Bernoulli measure of equation (8), and the three terms carry the following interpretation. Restricting to spatially uniform configurations ($\nabla_{\mathcal{C}}\psi = 0$) with $b = \frac{1}{4}$, the functional reduces to the compact scalar form:

$$\mathcal{F}(\psi) = -\varepsilon\,\psi^2 + \frac{1}{4}\,\psi^4, \qquad \varepsilon = |T(C)| - \tau \tag{23}$$

When $\varepsilon < 0$ (below threshold, precausal) $\mathcal{F}$ has a unique minimum at $\psi = 0$; when $\varepsilon > 0$ (above threshold, instantiated) two symmetric minima appear at $\psi^* = \pm\sqrt{2\varepsilon}$. A rescaled variant $-\frac{1}{2}\varepsilon\,\psi^2 + \frac{1}{4}\psi^4$ appears in the visualisations; its minima sit at $\psi^* = \pm\sqrt{\varepsilon}$, a factor of $1/\sqrt{2}$ smaller than $\pm\sqrt{2\varepsilon}$, but the bifurcation structure is preserved exactly. The full three-term structure of equation (22) is retained throughout the analysis.

The first term, $-\varepsilon(T(C))\,\psi^2$, governs the stability of the uninstantiated phase. The coefficient $-\varepsilon = \tau - T$ changes sign at $T = \tau$. When $T < \tau$, $\varepsilon < 0$ and this term increases $\mathcal{F}$ as ψ increases from zero, keeping the uninstantiated state $\psi = 0$ at a local minimum: configurations prefer to remain uninstantiated. When $T > \tau$, $\varepsilon > 0$ and the term now decreases $\mathcal{F}$ near $\psi = 0$, destabilising the uninstantiated state and driving the configuration toward nonzero ψ. The sign change in ε is precisely what defines the threshold.

The second term, $b\,\psi^4$ with $b > 0$, is the quartic stabilisation term. Without it, the functional would be unbounded below for $T > \tau$, and ψ could grow without limit. The quartic term ensures that once the uninstantiated phase becomes unstable, the configuration does not destabilise entirely but settles into a new, well-defined minimum at some finite value of ψ. It is this term that makes the transition a genuine phase change with a stable instantiated phase rather than a runaway.

The third term, $c\,|\nabla_{\mathcal{C}}\psi|^2$ with $c > 0$, penalises inhomogeneity in ψ across configuration space. If configurations with similar structures were to have very different modal statuses, the gradient would be large and the term would raise $\mathcal{F}$. This term is the formal analogue of the Ginzburg

gradient term and has the physical effect of producing smooth, coherent transitions rather than discontinuous patches of instantiated and uninstantiated configurations. Its presence ensures that the transition is second order rather than first order, a point treated below.

This functional is structurally identical to the Landau-Ginzburg free energy of second order phase transitions [8], and the analogy is illuminating. In condensed matter physics, the Landau-Ginzburg functional describes how a system moves between phases as temperature changes. The order parameter tracks the degree of order (magnetisation, superconducting condensate, and so on), and the free energy determines which phase is stable at which temperature. In PST, the role of temperature is played by the asymmetric tension $T(C)$, the role of the order parameter is played by the modal status ψ, and the potential determines which mode of being, uninstantiated or instantiated, is stable at which tension level.

The difference from the condensed matter case is fundamental. In condensed matter, the phase transition is a dynamical process: it unfolds over time, involves fluctuations and nucleation, and is driven by thermal agitation. In PST, there is no temperature, no heat bath, no temporal evolution, and no dynamics. The transition is a modal reorganisation: a change in which mode of being is logically available to the configuration, not a change in the state of a system that exists in time. The Landau-Ginzburg structure is borrowed for its mathematical economy, not for its physical interpretation.

4.5. Stability analysis and the two regimes

The most productive way to extract modal content from a functional is to identify which configurations minimise it. In classical mechanics, the principle of least action selects the physically realised trajectory from among all kinematically possible ones. In PST, the principle is modal rather than temporal: among all values of the order parameter available to a configuration of given tension $T(C)$, the modally self-consistent value is the one at which $\mathcal{F}$ attains its minimum. This is a variational statement, not a dynamical one. There is no time over which ψ evolves toward its minimum; the minimum is the only modal position a configuration can coherently occupy given its tension. Configurations whose order parameter deviates from the minimum of $\mathcal{F}$ are modally incoherent: their degree of instantiation is inconsistent with the tension they carry.

The condition for a minimum is the vanishing of the first functional derivative, $\delta\mathcal{F}/\delta\psi = 0$, the Euler–Lagrange condition applied to the modal functional. Before writing the result, it is worth identifying the structural role of each term in $\mathcal{F}$. The quadratic term $-\varepsilon\langle\psi,\psi\rangle$ is the modal restoring term: its coefficient $-\varepsilon = \tau - |T(C)|$ changes sign precisely at the threshold and is entirely responsible for the qualitative bifurcation in the behaviour of $\mathcal{F}$. The quartic term $b\langle\psi,\psi\rangle^2$ is the nonlinear saturation term: it is always positive and ensures the functional remains bounded below. The gradient term $c|\nabla_{\mathcal{C}}\psi|^2_{V_7}$ penalises rapid variation of ψ across

configuration space. Distributing $\delta/\delta\psi$ across these three terms and collecting, the modal equation of state is the V_7-vector equation:

$$-2\varepsilon\,\psi + 4b\,\langle\psi,\psi\rangle\,\psi - 2c\,\nabla^2_{\mathcal{C}}\psi = 0 \tag{24}$$

This equation has two qualitatively distinct regimes, separated by the threshold.

Below the threshold, $|T(C)| < \tau$: $\varepsilon < 0$. The only solution to the uniform equation is $\psi = 0 \in V_7$. Since $\delta^2\mathcal{F}/\delta\psi^2\big|_{\psi=0} = -2\varepsilon \mathbb{1}_{V_7} > 0$ (positive multiple of the identity), this is a local minimum in every direction of V_7. The configuration remains uninstantiated. This is the *precausal phase*.

Above the threshold, $|T(C)| > \tau$: $\varepsilon > 0$. The solution $\psi = 0$ is now a local maximum of $\mathcal{F}$ in every V_7-direction. The quadratic term falls with increasing $|\psi|$; the quartic term turns $\mathcal{F}$ upward again at large $|\psi|$. The result is a G_2-invariant Mexican-hat profile: a continuous family of degenerate minima at fixed radius, parameterised by the unit direction $\hat{\psi} \in S^6 \subset V_7$.

The location of these minima requires explicit computation. Restricting to spatially uniform configurations, the equation of state reduces to $-2\varepsilon\,\psi + 4b\,|\psi|^2\,\psi = 0$, i.e. $2\psi\,(-\varepsilon + 2b\,|\psi|^2) = 0$. The root $\psi = 0$ is the maximum. The nontrivial roots satisfy $|\psi|^2 = \varepsilon/(2b)$, giving the vacuum 6-sphere

$$\mathcal{V} \;=\; \{\,\psi \in V_7 \;:\; |\psi| = r_0\,\} \;\cong\; S^6, \qquad r_0 = \sqrt{\frac{|T(C)| - \tau}{2b}} = \sqrt{\frac{\varepsilon(C)}{2b}}. \tag{25}$$

Modal sublimation locks $\hat{\psi}$ to the substrate-determined direction $\hat{\tau} = T(D)/|T(D)|$ on $\mathcal{V}$ (P2). The earlier exposition's two "symmetric minima at $\psi = \pm r_0$" is the antipodal pair $\pm r_0\hat{\tau}$ in the 1-dim projection along $\hat{\tau}$; the full vacuum manifold has 5 additional directions on S^6 orthogonal to $\hat{\tau}$ that carry the gauge-theoretic content (§8.19). The appearance of $\varepsilon(C)$ under the radical is structurally significant. The depth of instantiation, how far from zero the order parameter settles, is not a free parameter but is determined directly by how far the configuration's tension sits above the threshold. A configuration with ε barely positive instantiates shallowly; one deep in the instantiated regime carries a large order parameter. This relationship between excess tension and the degree of instantiation is one of the central quantitative predictions of PST.

There are two degenerate minima, symmetric about $\psi = 0$. The configuration must occupy one of them, but the functional does not select which: both are equally deep. This is the *instantiated phase*. The two solutions correspond to the two structurally distinct orientations of the instantiated configuration, related by the symmetry $\psi \to -\psi$ that the functional possesses. That the functional carries this symmetry but the configuration that minimises it does not is the content of spontaneous symmetry breaking in the PST framework: the substrate selects a definite modal orientation without any external influence distinguishing the two options.

Structural versus spontaneous asymmetry: how P2 and P3 combine. The previous

paragraph affirms that PST has spontaneous symmetry breaking in the strict sense at the discrete Z_2 level: the $\psi \to -\psi$ symmetry of the functional is broken by the configuration's choice of one of two equally-deep minima. In the richer complex-scalar formulation (and *a fortiori* the full V_7 formulation), the LG potential also carries a continuous symmetry — a U(1) ring of degenerate minima in the complex slice, a G_2-orbit 6-sphere $\mathcal{V} \cong S^6 \subset V_7$ in the full setting — which in standard physics would be the canonical SSB setup producing Goldstone modes along $\mathcal{V}$.

PST departs from the standard SSB account at this angular level. P2's asymmetric vector tension supplies a structural preferred direction $\hat{\tau} = T(D)/|T(D)| \neq 0$ in V_7 *before any vacuum exists*, and modal sublimation *locks* $\hat{\psi}$ to $\hat{\tau}$ by direct determination from the substrate's directional bias (§1.7, P3 statement). The angular direction of the vacuum is therefore *not* a spontaneous choice: it is pinned by P2, upstream of the LG dynamics. What standard physics treats as a historical contingency — "which vacuum did the universe end up in?" — PST answers structurally: the substrate's pre-existing tension direction selects it.

Two complementary readings of this layering, both correct:

1. PST inherits the SSB *mathematics* — sombrero potential, degenerate vacuum manifold, Goldstone modes, Higgs radial mode, the EW eaten Goldstones at the doublet level (§8.4) — in full. The discrete Z_2 sign choice in the present paragraph is genuinely spontaneous.

2. PST *redirects* the spontaneous arbitrariness of the angular (continuous) direction: that selection is determined by P2, not by the SSB process. In standard-physics vocabulary the substrate's structural asymmetry is closer to an *explicit* symmetry breaking term, except that the symmetry was never present in the substrate's definition to begin with — P2 is not a perturbation added to a symmetric structure but a postulate of the substrate's structure itself.

The asymmetry is therefore not derived away in PST: it is relocated from the spontaneous-arbitrary level (where standard physics leaves it) to the structural-postulate level (P2). Whether this is progress depends on whether one finds the postulate that the substrate's tension functional is asymmetric under complementation more natural than a random vacuum selection at the SSB step.

The sign of a is therefore the fundamental discriminant between uninstantiated and instantiated existence. It is not a property of the configuration that can be read off directly; it is a property of the relationship between the configuration's tension and the threshold. This is why the threshold is primary and the transition is a logical consequence, not a dynamical one.

4.6. The bifurcation point and the derivation of τ

The critical tension magnitude τ is the value at which the qualitative character of $\mathcal{F}$ changes: for $|T(C)| < \tau$, the uninstantiated phase $\psi = 0 \in V_7$ is the unique minimum; for $|T(C)| > \tau$, it is a maximum and a 6-sphere of new minima appears. The moment of change is the **bifurcation point**, the magnitude of T at which $\psi = 0$ transitions from stable to unstable.

The characterisation of critical points as minima or maxima requires the second variation. For a V_7-valued field, the second functional derivative is a quadratic form on V_7 (a 7×7 symmetric tensor). If the form is positive-definite, the functional is locally convex at ψ_0 in every V_7-direction; the configuration is at a stable minimum. If indefinite or negative-definite, at least one direction in V_7 is unstable.

The critical point of interest is $\psi = 0$, the uninstantiated configuration. To determine whether this is modally stable (the precausal phase is robust) or unstable (it cannot be maintained), one must examine the curvature of $\mathcal{F}$ at this point. Differentiating $\delta\mathcal{F}/\delta\psi = -2\varepsilon\psi + 4b\,|\psi|^2\psi$ once more with respect to ψ and evaluating at $\psi = 0$ eliminates the cubic-in-ψ terms entirely. What remains is purely the contribution of the quadratic term, proportional to the identity on V_7:

$$\left.\frac{\delta^2\mathcal{F}}{\delta\psi^2}\right|_{\psi=0} = -2\varepsilon(|T|)\,\mathbb{1}_{V_7} = 2\,(\tau - |T|)\,\mathbb{1}_{V_7} \tag{26}$$

This is positive-definite for $|T| < \tau$ (i.e., $\varepsilon < 0$), zero for $|T| = \tau$, and negative-definite for $|T| > \tau$ (i.e., $\varepsilon > 0$). The G_2-isotropy of the second variation at $\psi = 0$ is direct: the form is a scalar multiple of the identity, hence invariant under all V_7-rotations. The second variation vanishes at $T = \tau$, and this vanishing is the mathematical realisation of the threshold: the precise tension value at which the uninstantiated configuration ceases to be stable. What remains is to give τ a rigorous definition that does not presuppose its own value.

The naive approach would be to define τ as the solution to $\delta^2\mathcal{F}/\delta\psi^2\big|_{\psi=0} = 0$, which from the expression above gives $-2\varepsilon = 0$ and thus $|T| = \tau$, a circularity that conceals rather than reveals. The correct approach is to observe that τ is defined by a property of the functional, not by an algebraic equation: it is the magnitude below which $\psi = 0$ is a stable minimum and above which it is not. This is precisely the infimum of the set of tension magnitudes for which the second variation vanishes:

$$\tau = \inf\left\{|T(C)| : \left.\frac{\delta^2\mathcal{F}}{\delta\psi^2}\right|_{\psi=0} = 0\right\} \tag{27}$$

The use of the infimum rather than a simple equation reflects the possibility that configurations at exactly $T = \tau$ form a set of measure zero in $\mathcal{C}$: the threshold is the greatest lower bound of all tension values that can trigger instantiation, and this bound may or may not be achieved by any individual configuration. In practice, when the functional takes the quadratic-quartic

form above, the infimum is achieved and τ is well-defined as a specific value.

The significance of this derivation is that τ emerges from the structure of $\mathcal{F}$ itself. It is not a parameter that the theorist chooses. Once the functional is specified, and its form is constrained by the symmetries of the precausal substrate and the requirement that the transition be second order: the threshold is fixed. In this sense, the modal threshold is not a contingent feature of the theory. It is a structural necessity: any substrate governed by a modal potential of this form will have a threshold, and that threshold will be the bifurcation point of the functional.

The passage from $\psi = 0$ to the bifurcated minimum is modal sublimation, with no temporal unfolding. The next section characterises that transition in detail.

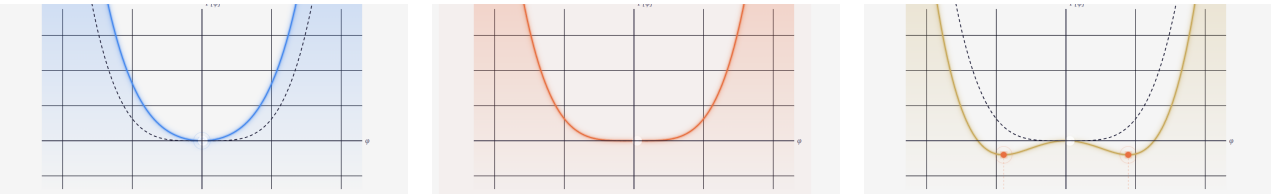

Figure 1: The modal potential $F[\varphi]$ at three values of excess tension ε. *Left:* $\varepsilon = -0.8$, single valley at $\varphi = 0$; the substrate configuration rests at the unique minimum, uninstantiated. *Centre:* $\varepsilon = 0$, the modal threshold; the valley flattens and the minimum is about to split. *Right:* $\varepsilon = +0.8$, two degenerate minima at $\varphi = \pm\sqrt{2\varepsilon}$; the configuration at $\varphi = 0$ is now a local maximum and cannot remain there. The dashed curve is the reference potential at $\varepsilon = 0$. The four key relations carried by this figure are: (1) the Landau-Ginzburg modal potential $F(\psi, \varepsilon) = -\varepsilon \langle \psi, \psi \rangle + \frac{1}{4}\langle \psi, \psi \rangle^2$ with quartic coefficient $b = \frac{1}{4}$ fixed by the LG form; (2) excess tension $\varepsilon = |T(\mathcal{C})| - \tau$; (3) the vacuum manifold radius $|\psi^*| = \sqrt{2\varepsilon}$ on V_7 with topology $\mathcal{V} \cong S^6$ (a 6-sphere of degenerate directions in V_7, projecting onto S^1 in any 2-plane slice of V_7); and (4) the projection map $\Phi : \mathcal{S} \to \mathcal{G}$ from the precausal substrate to instantiated geometry.

4.7. Summary of the threshold half

Not every tension imbalance produces a universe. This section derives a critical tension value, the modal threshold τ, below which the substrate remains abstract and unobservable, and above which it cannot remain that way. Think of water vapour: it can exist invisibly up to a certain temperature, but above that point it must condense. Below τ, the configuration has a single stable resting state with no geometry; above τ, that resting state becomes unstable and geometry must appear. The threshold is not assumed: it is calculated as the exact tension value at which the mathematical potential changes from single-welled to double-welled, and its existence is a structural necessity rather than a postulate.

Claim status of § 3. *Postulates*: P2 (asymmetric vector tension; $T(D) \neq 0$ in V_7 generically) and P3 (the G_2-invariant Landau-Ginzburg modal potential on V_7). *Derived theorems*: existence and value of the modal threshold τ as the bifurcation point of $F[\psi, \varepsilon]$; vacuum manifold $\mathcal{V} \cong S^6 \subset V_7$ for $\varepsilon > 0$. *Structural disclosure (§ 3.x)*: at the angular level the vacuum direction

is *not* a spontaneous choice; it is pinned by P2's $\hat{\tau}$. At the discrete Z_2 level the SSB is genuinely spontaneous in the standard sense. These layered roles are made explicit in the "structural versus spontaneous asymmetry" paragraph. *Modelling choices*: G_2-invariant quartic functional form; the Bernoulli measure of § 2 as the configuration-space measure under which F is varied.

4.8. The character of the transition

The transition induced by crossing the modal threshold is unlike any physical process encountered in the instantiated world. It has no duration, no mechanism, no intermediate states, and no energy cost. It is not that the transition is very fast; temporal concepts simply do not apply to it. The precausal substrate is a nonspatiotemporal domain: there is no time in which a transition could unfold, and no space across which it could propagate. What occurs at the threshold is not a process but a **modal reorganisation**, a shift in the mode of being of the configuration from uninstantiated to instantiated.

This transition is called **modal sublimation**, by analogy with the physical phenomenon in which a solid passes directly into a gaseous state, bypassing the liquid phase. In modal sublimation, every intermediate mode is bypassed because only two modes of being are available: uninstantiated existence within the precausal substrate, and instantiated existence as geometry. There is no intermediate mode that is partially spatial or partially causal.

Modal sublimation is characterised by four properties:

- **Nontemporal**: no before or after, no duration, no sequence of states; the configuration transitions *into* time rather than *through* it
- **Nonmechanistic**: no propagating influence, no force, no interaction
- **Nonenergetic**: no conservation law, no energy input or output
- **Strictly modal**: a change in the mode of being, not a change of state within an existing mode

Each of these four properties requires unpacking, because each contradicts a default assumption carried over from the instantiated world.

4.9. The nontemporal character

The nontemporal character is the most radical. In everyday physics, every process unfolds in time: there is a before and an after, a cause and an effect, a duration that can in principle be measured. Even processes described as instantaneous in non-relativistic mechanics, a collision, a decay, a measurement, are idealised limits of processes that have a finite, if arbitrarily small, duration. Modal sublimation is not a limit of this kind. It does not occur quickly; it does not occur in time at all. The configuration does not spend even an infinitesimal moment in transition. The reason is not dynamical but ontological: time is a structure of the instantiated

domain, and the transition is the very event that brings that domain into being. To ask how long modal sublimation takes is to apply a concept to a context in which the concept has not yet been defined. This is not a failure of measurement precision; it is a logical impossibility of the same kind as asking what temperature a mathematical proof achieves. The configuration does not transition *through* time; it transitions *into* time, carrying time as a consequence of instantiation rather than as a pre-existing container in which the transition unfolds.

4.10. The nonmechanistic character

The nonmechanistic character follows directly from the nontemporal character. In the instantiated domain, every change of state is mediated by an interaction: a force, a field, a messenger particle, a propagating influence of some kind. These mediators all require spacetime in which to propagate and physical degrees of freedom to couple to. The precausal substrate has neither. No interaction can mediate the modal sublimation because interactions are structures of the domain that sublimation brings into being. There is no force that pushes a configuration across the threshold; the threshold is not a barrier that must be overcome but a logical boundary at which a particular mode of being ceases to be available. The transition is not driven by anything external. It is a logical consequence of the modal structure of the configuration itself, in the same way that the truth of a mathematical theorem is not caused by anything but follows necessarily from the axioms. This has a crucial implication: PST does not require a mechanism, an initial impetus, or a first cause. The demand for a mechanism is itself an instantiated-world concept, and applying it to the transition that precedes instantiation is a category error of the same kind as the causal question discussed in Chapter 5.

4.11. The nonenergetic character

The nonenergetic character removes the last temptation to treat modal sublimation as a physical event in disguise. Energy is a conserved quantity defined within the instantiated domain: it requires a Hamiltonian, a time-translation symmetry via Noether's theorem, and ultimately a spacetime manifold on which these structures can be defined. None of these are available in the precausal substrate. The relevant conservation statement in PST is not energetic but modal: it is the preservation of the asymmetric tension functional $T(C)$ across the threshold. The tension that characterises the precausal configuration is the same quantity that, after projection via Π, appears as the stress-energy tensor $T_{\mu\nu}$ in the instantiated domain, as shown in Chapter 7. The Einstein field equations, on this reading, are not equations of motion governing a physical process; they are a **conservation statement** expressing that the structural content of the precausal configuration is faithfully preserved in the instantiated geometry. Nothing is created across the threshold; nothing is destroyed. The same structural content is expressed in a different modal register, the way that the same logical argument can be expressed in different formal languages without gaining or losing content.

4.12. The strictly modal character

The strictly modal character is the most philosophically precise formulation of what the other three properties imply. A change of state *within* an existing mode, a particle moving, a field oscillating, a temperature rising, leaves the mode of being intact while altering something within it. The ontological register is unchanged: the system is still a physical system in spacetime, governed by the same causal laws, measured in the same units. Modal sublimation changes the mode itself. The configuration does not acquire new properties within the precausal substrate; it changes the ontological register in which all its properties are expressed. Before sublimation, the configuration's structural content exists as modal tension in a nonspatiotemporal domain. After sublimation, that same structural content exists as geometric curvature and matter-energy in a Lorentzian manifold. There is no moment at which the configuration is partially in each mode; the transition is all-at-once in the modal sense, even though "all-at-once" is itself a temporal phrase that strictly applies only after time has come into being. This is the precise sense in which PST's fundamental transition is irreducible to any physical process: physical processes are changes of state within the instantiated mode, and modal sublimation is the event that instantiates the mode itself.

The result of this transition is a **geometry**: a Lorentzian manifold (M, g) that becomes the spacetime background for all subsequent physical processes. Geometry does not emerge *within* spacetime; it *becomes* spacetime in the act of sublimation.

4.13. Categorical formulation

The mathematical language suited to a transition between fundamentally different types of structure is category theory, specifically the theory of functors between categories [9]. Construct two categories as follows. The **precausal category S** has as objects all configurations $C \in \mathcal{C}$ with $T(C) \geq \tau$, and as morphisms distinction preserving maps $f : C \to C'$. The **geometric category G** has as objects Lorentzian manifolds (M, g) and as morphisms smooth maps preserving the causal structure.

Modal sublimation is the functor:

$$\Phi : \mathbf{S} \to \mathbf{G} \tag{28}$$

with four properties encoding the fundamental features of the transition:

1. Φ **exists at the categorical boundary**; it is not a morphism within **S** or **G**, formalizing the nonmechanistic character of the transition
2. Φ **preserves tension as curvature**: $T(C) \mapsto \kappa(\Phi(C))$, where κ is the scalar curvature of the instantiated manifold
3. Φ **is surjective**: every Lorentzian manifold arises as the image of some precausal configuration

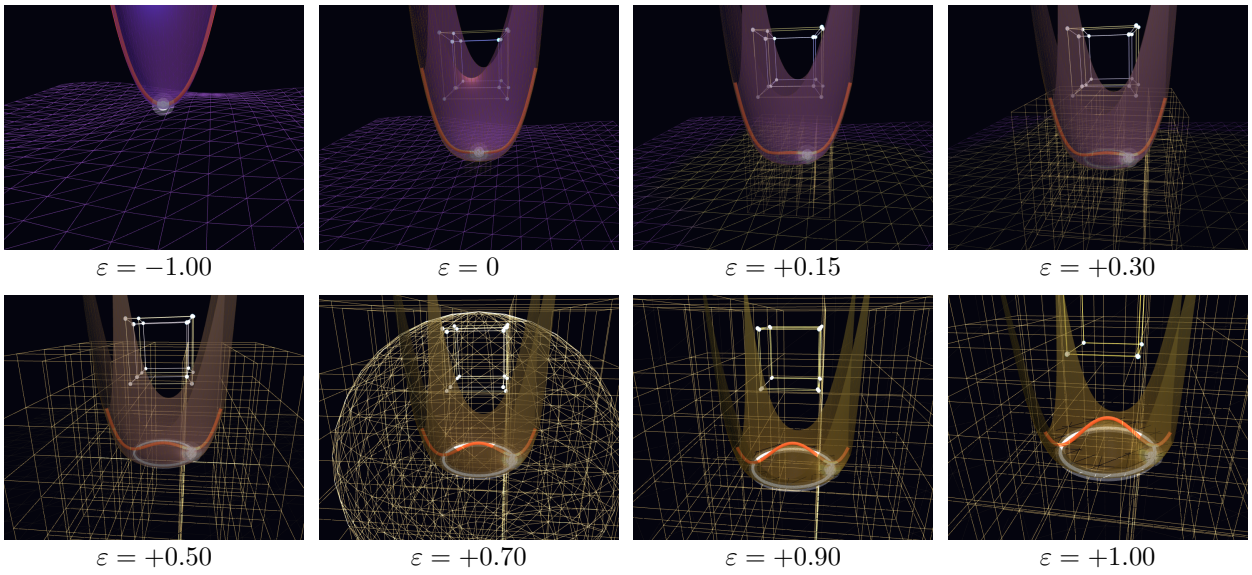

Figure 2: Visual model of modal sublimation across eight values of the excess tension $\varepsilon = |T(C)| - \tau$ (Chapter 4): $\varepsilon < 0$ precausal, $\varepsilon \geq 0$ instantiated. Sampling is dense on the instantiated side, where the geometric content emerges; a single precausal reference frame at $\varepsilon = -1.00$ shows the pre-threshold state from which everything else unfolds. Each frame shows the modal potential $F[\psi, \varepsilon] = -\varepsilon\psi^2 + \frac{1}{4}\psi^4$ rendered as a rotationally symmetric surface (lower portion), with the orange tube tracing its one-dimensional cross-section and the white sphere marking the position of the order parameter ψ at the current potential minimum. *Top-left frame* ($\varepsilon = -1.00$): the precausal substrate; the potential forms a deep bowl with a single minimum at $\psi = 0$, the order parameter is confined to the origin, and a flat proto-manifold grid is shown for reference. *Threshold and early instantiation* ($\varepsilon = 0$ to $\varepsilon \approx +0.30$): the bowl flattens and its centre rises, the high-dimensional cube descends toward the bowl, the spacetime lattice crystallises, and the ring of minima begins to form at the base. *Mid to deep instantiation* ($\varepsilon \approx +0.50$ to $+1.00$): the potential has acquired a continuous 6-sphere of degenerate minima (the vacuum manifold $\mathcal{V} \cong S^6 \subset V_7$); the figure displays the 2-plane projection visible as an orange ring $\mathcal{V}_{\hat{\eta}} \cong S^1$ in any $\hat{\tau}$-perpendicular slice; the spatial slice closes, shown by the S^3 dome enveloping the local chart; and the finite internal factor $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ consolidates as the cube snaps onto the spacetime grid.

4. Φ **has no inverse**: once a configuration is instantiated, the geometric domain has no access to the precausal substrate that produced it

4.14. Explicit construction of Φ

The projection functor Φ inherits the post-sublimation directional structure of the substrate: it is constructed *relative to* the $\hat{\tau}$-direction that modal sublimation locks (P2). The construction below produces a spacetime M and a realisation ρ together with an internal-factor identification along $\hat{\tau}$; the same construction at a different $\hat{\tau}' \in S^6$ would produce a G_2-rotated copy of the same Φ, with the rotation an unphysical relabelling. Once $\hat{\tau}$ is fixed by the substrate's directional bias, Φ is determined up to the gauge freedom described below.

The functor is constructed in three steps.

Step 1, Geometric realization. The manifold M and the map ρ are not given independently before the construction; they are jointly determined by it.

A legitimate concern requires attention before the construction proceeds. The notation $\rho : D \to M$ appears to presuppose M, a set of spacetime points, before the construction that is supposed to *produce* spacetime has been carried out. If spacetime points are already available as the target of ρ, then the theory has not derived spacetime from the substrate; it has merely labeled pre-existing points with properties from D. This would be a foundational circularity.

The resolution is that M is not given prior to and independently of ρ. The construction proceeds in the following order. First, note that any surjective function $\rho : D \twoheadrightarrow X$, for *any* set X, partitions D into equivalence classes: $a \sim_\rho b$ if and only if $\rho(a) = \rho(b)$. The equivalence classes $[a]_\rho = \{b \in D : \rho(b) = \rho(a)\}$ are the fibres of ρ, and their collection $D/\!\sim_\rho$ is a quotient set that depends only on D and the partition, not on any property of X. A spacetime *point* is then defined to be a fibre of ρ: it is an equivalence class of properties, not a pre-existing geometric entity. The manifold M is the quotient set $D/\!\sim_\rho$ equipped with whatever additional structure (topology, differentiable structure, metric) the subsequent steps impose. The map ρ is thus a *quotient map*, and M is a *derived* set: it comes into existence as the image of ρ, rather than being a container that ρ maps D into.

This construction has a philosophical precedent in Leibniz's principle of the identity of indiscernibles: things that share all the same properties are the same thing. Here, the converse is operative, properties that are assigned to the same geometric locus by ρ are identified, and the locus is nothing over and above that identification. A spacetime point has no content beyond the property-bundle it collects; it is not an additional ontological entity.

More concretely: M is characterised as *any* smooth manifold of dimension n whose topology and differential structure are consistent, a posteriori, with requiring that the metric constructed in Step 2 is well-defined and smooth. Among all candidate manifolds satisfying this consis-

tency condition, the tension structure of (D, δ) further constrains which are admissible: a candidate M is admissible if and only if there exists a partition of D whose induced tension structure $T(\{[a], [b]\}) = T(\{a, b\})$ reproduces the relational data already present in (D, δ). This constraint, not an external stipulation, selects the spacetime topology and dimension.

Given a choice of admissible M, define the **realization map** $\rho : D \to M$ as the quotient projection sending each property $a \in D$ to its equivalence class $\rho(a) = [a]_\rho \in M$. The nonuniqueness of ρ, the freedom to partition D into fibres in multiple ways that are all consistent with the tension structure, is precisely the diffeomorphism invariance of General Relativity [1, 10]. It emerges here as a consequence of the nonuniqueness of geometric realization from the pre-geometric substrate, rather than as a separately imposed symmetry. There is no canonical labeling of spacetime points: this is not a deficiency of the theory but a structural result, matching the gauge freedom of GR.

Step 2, Tension-to-metric map. Introduce the **null threshold** $\tau_{\text{null}} \in \mathbb{R}^+$ as a second structural input alongside τ. While τ is a condition on the global tension $T(C)$, τ_{null} is a condition on pairwise tensions $T(\{a, b\})$: it is the value at which the two-property tension is null, i.e., at which $g = 0$ by definition. The two thresholds operate at different levels. Their relationship reduces to the same open structural input as the rate-integral computation of §11.7: an explicit form of the global tension functional $T(C)$ as a functional of the pairwise tensions $T(\{a, b\})$ at the postulate level. Once the explicit functional is specified, τ is the critical value of $T(C)$ at which the modal potential loses its single-well stability (§4.1) and τ_{null} is its pairwise restriction; the two are then determined jointly, not independently. The metric is then:

$$g_{\mu\nu}(\rho(a), \rho(b)) = \eta_{\mu\nu} \cdot \operatorname{sgn}(T(\{a, b\}) - \tau_{\text{null}}) \cdot |T(\{a, b\}) - \tau_{\text{null}}|^{1/2} \tag{29}$$

where $\eta_{\mu\nu}$ is a placeholder for the diagonal sign pattern $\operatorname{diag}(-1, +1, +1, +1)$ that the construction induces, not a presupposition of Minkowski geometry. Its appearance here is definitional: the signature theorem below shows that the sign pattern is fixed by the tension structure; $\eta_{\mu\nu}$ names that pattern. This gives:

- $T(\{a, b\}) = \tau_{\text{null}} \Rightarrow g = 0$: null separation
- $T(\{a, b\}) > \tau_{\text{null}} \Rightarrow g > 0$: spacelike
- $T(\{a, b\}) < \tau_{\text{null}} \Rightarrow g < 0$: timelike

The Lorentzian signature is forced as a theorem by hyperbolic well-posedness of the Goldstone wave equation (Computation 2, §13.10): the Cauchy problem for $\Box_g \theta = 0$ admits a well-posed initial-value formulation and a conserved Noether current only for signature $(1, n-1)$. The bifurcation of tension values around τ_{null} remains the algebraic mechanism inside the LG variational structure; what Computation 2 shows is that the sign cut required for that bifurcation is not an independent axiom but a consequence of the well-posedness condition.

Two clarifications are necessary. First, equation (29) is strictly a pairwise assignment: it attaches a signed scalar to each pair of realised points, not a $(0,2)$-tensor to a tangent space. The metric tensor field $g_{\mu\nu}(x)$ is recovered by the continuum limit: take a and b to be nearby properties such that $\rho(a)$ and $\rho(b)$ approach x along a tangent vector v^μ; then $g_{\mu\nu}(x)\, v^\mu v^\nu$ arises as the limit of equation (29) along the chord, analogous to recovering a Riemannian metric as the Hessian of a distance function. Second, the magnitude $\left|T(\{a,b\}) - \tau_{\text{null}}\right|^{1/2}$ is the minimal-regularity choice: a linear dependence would give a metric with a non-smooth gradient at the null cone, while the square root yields Hölder-$\frac{1}{2}$ behaviour there, the appropriate regularity class for a structure that degenerates at $T = \tau_{\text{null}}$ and which ensures that equation (185) blows up precisely at the modal null threshold rather than at $T = 0$.

Step 3, Action on morphisms. For $f : C \to C'$ in $\mathbf{S}$, $\Phi(f) : (M, g) \to (M', g')$ is the diffeomorphism induced by f via ρ. Distinction preserving maps in $\mathbf{S}$ correspond precisely to causal structure preserving maps in $\mathbf{G}$, ensuring functoriality.

The projection operator. Define the **tension projection operator** Π:

$$\Pi(T(C))(x) = \int_{\mathcal{C}} T(C') \cdot K(x, C')\, d\mu(C') \tag{30}$$

where $K(x, C') = \delta(x - \rho(C'))$ for local configurations. The role of Π here is structurally analogous to the Dirac operator in Connes' noncommutative geometry [48]: in both cases a single operator encodes the geometric and algebraic content of the instantiated domain from an underlying non-spatial structure. The stress-energy tensor at $x \in M$ in the isotropic, homogeneous case is:

$$T^{(\text{iso})}_{\mu\nu}(x) = \Pi(T(C))(x) \cdot g_{\mu\nu}(x) \tag{31}$$

This isotropic scalar form $T^{(\text{iso})}_{\mu\nu} = \Pi(T(C))\, g_{\mu\nu}$ gives a stress-energy tensor proportional to the metric, which is the correct structure for a cosmological constant ($T_{\mu\nu} = \Lambda g_{\mu\nu}$) but not for general matter distributions. Pressurised matter, radiation, and anisotropic fields all have $T_{\mu\nu} \neq \lambda g_{\mu\nu}$ for any scalar λ. The general form requires Π to be a tensor-valued projection $T_{\mu\nu}(x) = \Pi_{\mu\nu}(T(C))(x)$, whose components encode the directional distribution of tension in the precausal configuration. The scalar form is the leading-order approximation valid for isotropic configurations; the full tensor generalisation is an open problem. The noninvertibility of Φ is immediate: ρ discards the internal structure of D that does not affect pairwise tensions. Geometry encodes the projection of T through Π but not the full precausal configuration that generated it, just as a shadow records an object's outline but not its internal structure.

4.15. Uniqueness of Φ and the origin of diffeomorphism invariance

The construction proceeds via an explicit choice of realization map $\rho : D \to M$, which is not determined by the precausal structure alone. This raises the question of how strongly the

output functor Φ depends on the choice of ρ, and what that dependence implies for the physical content of the theory. The answer is a uniqueness theorem: Φ is unique up to diffeomorphism, and the ambiguity in Φ is completely and exactly characterised by diffeomorphism invariance, with no residual freedom beyond it.

The proof follows directly from the construction. Given ρ, the metric $g_{\mu\nu}$ at every pair of points is fully determined by equation (29): there are no further choices once the realization map is fixed. Now suppose $\rho' = f \circ \rho$ for some smooth bijection $f : M \to M'$. The tension values $T(\{a, b\})$ depend only on $a, b \in D$, not on their images under ρ, so they are unchanged by the substitution. Therefore the metric computed at $(\rho'(a), \rho'(b))$ via equation (29) satisfies $g'(\rho'(a), \rho'(b)) = g(\rho(a), \rho(b))$, which is exactly the condition that (M', g') is the pushforward of (M, g) along f. Any two choices of ρ therefore produce Lorentzian manifolds that are diffeomorphic to one another, together with a corresponding natural isomorphism between the two functors they define. Φ is unique up to this natural isomorphism, which at the level of manifolds is a diffeomorphism.

The physical significance of this result is that diffeomorphism invariance in General Relativity is not an additional symmetry requirement that must be imposed upon the theory by hand. It is the complete characterisation of the non-uniqueness of Φ: the relabelling freedom in choosing ρ survives into the instantiated manifold as coordinate freedom, and coordinate freedom is diffeomorphism invariance. There is no finer ambiguity and no coarser one; the two are identical. The natural question of whether Φ is unique and what physical symmetries characterise its ambiguities is therefore answered here: Φ is unique up to diffeomorphism, its ambiguity is completely characterised, and the physical symmetry corresponding to that ambiguity is diffeomorphism invariance.

A note on the metric signature. The argument in Step 2 assigned the Minkowski signature $(+, -, -, -)$ through equation (29), by bifurcating pairwise tensions about a null threshold τ_{null} that separates spacelike from timelike separations. This determines the signature *given* τ_{null}: once a sign cut is supplied, $T > \tau_{\text{null}}$ is spacelike and $T < \tau_{\text{null}}$ is timelike, by construction. It does *not* determine that exactly one direction lies below τ_{null}, spectral asymmetry of the tension spectrum about a threshold (the standard form of which is $T(C) \neq T(\bar{C})$, complement-asymmetry) is consistent with zero, one, or several eigenvalues lying below any given cut, so it does not by itself fix the count of timelike directions. Equivalently: in the multidimensional-scaling reframing of the realisation map (§13.10), any S_D-invariant Gram construction over real features is positive-semidefinite by elementary linear algebra, so the negative eigenvalue required for one timelike direction cannot arise from S_D-symmetry alone; it must be supplied by an indefinite bilinear form imposed by hand. The Lorentzian signature $(+, -, -, -)$ is fixed as a *theorem* via Computation 2 of §13.10: hyperbolic well-posedness of the Goldstone wave equation forces $(p, q) = (1, n-1)$, and the U(1) Noether-current conservation that produces the angular-momentum identification requires this same Lorentzian signature. The macroscopic

dimension $n = 4$ that combines with the signature to give the four-dimensional Lorentzian spacetime $M = \mathbb{R} \times S^3$ follows from KO additivity given the verified spectral-triple structure (§1.5): the substrate's parity-selected KO-6 (Computation 3) and the product's KO-2 mod 8 (Connes Lorentzian value) together force M at KO-dimension four. PST is then the theory that derives everything downstream from (D, δ) + LG + the realisation map ρ, with both signature and macroscopic dimension outputs rather than inputs.

4.16. Lorentzian signature from Goldstone hyperbolic well-posedness

Earlier passages referenced "Lorentzian signature follows from hyperbolic well-posedness of the Goldstone wave equation (Computation 2)." This subsection makes the argument self-contained: we state the PDE, the function spaces, the energy estimate, and show why one timelike direction (signature $(p, q) = (1, n-1)$) is the unique signature compatible with a well-posed Cauchy problem on the projected spacetime $\mathbb{R} \times S^3$.

The PDE. After modal sublimation has selected the vacuum manifold $\mathcal{V} \cong S^6 \subset V_7$ (§1.7, P3), the Goldstone phase θ along the broken direction satisfies, at the quadratic order of the LG functional, the second-order PDE

$$g^{\mu\nu}\,\partial_\mu\partial_\nu\,\theta(x) \;=\; 0, \qquad x \in M, \;\; \mu, \nu = 0, 1, \ldots, n-1, \tag{32}$$

where $g^{\mu\nu}$ is the (Wick-rotation-undetermined) signature-(p, q) metric on $M = \mathbb{R} \times S^3$ obtained as the projection of the LG kinetic term to emergent spacetime. The order parameter contributes $r_0^2|\nabla\theta|^2$ to the LG free energy, which under $\nabla \to g^{\mu\nu}\partial_\mu \otimes \partial_\nu$ becomes the wave operator.

Cauchy problem and function spaces. The Cauchy problem for equation (32) on the time slice $\Sigma_0 = \{0\} \times S^3$ asks: given initial data $(\theta_0, \theta_1) \in H^1(S^3) \times L^2(S^3)$, does there exist a unique solution $\theta \in C(\mathbb{R}; H^1(S^3)) \cap C^1(\mathbb{R}; L^2(S^3))$ with $\theta|_{\Sigma_0} = \theta_0$, $\partial_t\theta|_{\Sigma_0} = \theta_1$?

For the Cauchy problem to be *well-posed in the Hadamard sense* (existence, uniqueness, and continuous dependence on initial data) the principal symbol $g^{\mu\nu}\xi_\mu\xi_\nu$ of equation (32) must be *strictly hyperbolic* with respect to the splitting Σ_0, i.e. the polynomial

$$g^{00}\tau^2 \;+\; 2g^{0i}\tau\xi_i \;+\; g^{ij}\xi_i\xi_j \;=\; 0 \tag{33}$$

must have exactly two real roots $\tau = \tau_\pm(\xi)$ for every nonzero spatial covector $\xi \in T^*\Sigma_0$ [112, 113, 99]. Strict hyperbolicity forces the metric to have one timelike direction (transverse to Σ_0) and $n-1$ spacelike directions. Equivalently, signature $(p, q) = (1, n-1)$ (or its sign-reversal $(n-1, 1)$, the same content).

Energy estimate. Given strict hyperbolicity, multiplying equation (32) by $\partial_t\theta$ and integrating

over Σ_t yields the standard wave energy

$$E(t) \;=\; \tfrac{1}{2}\int_{\Sigma_t} \big((\partial_t\theta)^2 \;+\; g^{ij}\partial_i\theta\,\partial_j\theta \big)\sqrt{g_{S^3}}\,dx \tag{34}$$

which is conserved: $E(t) = E(0)$ for all $t \in \mathbb{R}$. Energy conservation immediately gives existence (via the standard semigroup construction [99]), uniqueness (the zero-data Cauchy problem has zero energy and hence zero solution by Gronwall), and continuous dependence (initial-data perturbations propagate at finite speed bounded by $\max|g^{ij}/g^{00}|^{1/2}$ along light cones). This is the standard Sobolev-space well-posedness theory for the wave equation [112, 113].

What fails in other signatures. Three alternatives must be considered and ruled out:

1. *Euclidean* $(p, q) = (0, n)$. All directions are spacelike; the principal symbol $g^{\mu\nu}\xi_\mu\xi_\nu$ is positive-definite. Equation (32) becomes *elliptic* (Laplace) rather than hyperbolic. The Cauchy problem for an elliptic equation is famously *ill-posed* (Hadamard's classical example [109]): small perturbations in the data give exponentially growing perturbations in the solution, so no continuous-dependence map exists. Modal sublimation produces a domain that admits time evolution, not an elliptic boundary-value problem; Euclidean signature is therefore incompatible with the Cauchy-problem structure of modal sublimation.

2. *Ultra-hyperbolic* (p, q) *with* $p \geq 2$, $q \geq 2$. Two or more timelike directions. The principal symbol has signature (p, q) as a quadratic form, but the Cauchy problem on the codimension-p "time" submanifold has multiple time derivatives in the data, and strict hyperbolicity in the sense above fails: the polynomial in τ has either no real roots (for some ξ) or more than two. The Cauchy problem is ill-posed in the Hadamard sense; existence fails for generic smooth data [110, 111].

3. $(p, q) = (n, 0)$ *(anti-Lorentzian).* All directions timelike, no spatial direction. Equation (32) has no propagation direction; the projection has no instantiated geometry to speak of. Excluded as a degenerate case.

Conclusion. The unique signature compatible with a well-posed Cauchy problem for the Goldstone wave equation on the Mosco-limit spacetime $M = \mathbb{R} \times S^3$ is $(p, q) = (1, n - 1)$. Combined with the $n = 4$ fix from the minimal KO-split (§1.5), this gives $(p, q) = (1, 3)$, i.e. signature $(+, -, -, -)$. Lorentzian signature is therefore a *theorem* of modal sublimation rather than an additional postulate: the directionality of the threshold crossing selects time, hyperbolic well-posedness selects the number of spatial directions, and Computation 2 verifies the explicit energy-estimate construction numerically on representative finite models.

Status. The above is a structural derivation using standard PDE theory [112, 113, 99]; the energy estimate is rigorous and the alternative-signature arguments are sketched at standard textbook level rather than fully worked. A fully rigorous treatment with detailed Sobolev-space analysis is straightforward but would expand the appendix substantially; the structural

conclusion does not depend on which level of analytic detail one prefers. Computation 2 supplies the numerical verification of energy conservation on a finite-mode truncation of S^3.

4.17. Summary

Modal sublimation is the moment the threshold is crossed, not a moment in time, but a logical transition. When a configuration's tension exceeds τ, it cannot remain in its abstract precausal state; it is compelled, by the same kind of necessity that makes a mathematical theorem true, to become concrete. The result is a four-dimensional spacetime with the Lorentzian signature that physics recognises. This section constructs the map from the precausal domain to the geometric domain, shows it is unique, and proves that the coordinate-freedom of General Relativity (diffeomorphism invariance) is a consequence of that uniqueness rather than a separate postulate.

Claim status of § 4. *Construction*: the modal-sublimation functor $\Phi : \mathcal{S} \to \mathcal{G}$ is constructed explicitly in three steps. *Derived (Computation 2)*: hyperbolic well-posedness of the Goldstone wave equation along $\hat{\tau}$ forces Lorentzian signature $(+,-,-,-)$. The energy-estimate and PDE details are sketched here; an expanded self-contained derivation is the subject of a planned v23.x revision. *Modelling choice*: τ_{null} is the sign cut where hyperbolicity flips; it follows from the requirement of well-posedness rather than being posited. *Derived*: diffeomorphism invariance from the non-uniqueness of the realisation map ρ. *Conditional theorem (Mosco convergence)*: $\mathcal{S} \to M = \mathbb{R} \times S^3$ rests on Mosco convergence of the Boolean Dirichlet form to the Laplace-Beltrami form on M. The argument is structural; the conditional theorem is stated explicitly in §7.7 with six explicit assumptions A1–A6, and the equicoercivity assumption A6 – previously flagged as the central open analytic item – is reduced in §7.7 to a uniform-rate hypothesis (R) on H^1-bounded compactly-supported test functions plus a standard Poincaré inequality, with the remaining K-uniform Berry-Esseen-type rate identified as standard finite-element analysis territory (Brenner-Scott [101], Computation 70). *Modelling choice (extremal)*: the minimal KO-split $4 + 6 = 10$ given KO-6 substrate (§1.5) fixes $n = 4$.

5. Coinstantiation of Spacetime and Causality

5.1. Causality as coinstantiated structure

The previous section established that modal sublimation is a nontemporal, nonmechanistic transition: the functor $\Phi : \mathbf{S} \to \mathbf{G}$ maps a precausal configuration directly into a Lorentzian manifold without passing through any intermediate state and without requiring a prior temporal structure in which the transition could unfold. A consequence of this that deserves its own careful treatment is the status of causality itself. Causality is not, in PST, a background structure that was already present when the universe formed and that can then be applied to the event of its formation. Causality is **coinstantiated** with geometry: it arises at the same

moment as the Lorentzian manifold that carries it, because it is constitutively defined in terms of that manifold. There is no moment, however brief or abstract, at which a causal relation exists but spacetime does not. The two are ontologically inseparable.

This point carries weight precisely because it is not obvious. Much of the philosophical and cosmological literature treats causality as prior to or more fundamental than spacetime. In the Kantian tradition, causality is a category of the understanding applied to experience, logically prior to any empirical fact about the world [34]. In causal set theory, the discrete partial order is the primary structure from which the continuum metric is to be recovered [4, 35]. In quantum gravity approaches that proceed via sum over histories, the causal structure of each history is taken as given, and the amplitude is computed over those histories [36]. Even in ordinary language, the question "what caused X?" is considered answerable in principle for any event X, the presumption being that every event has a cause and that the causal relation is always available to be invoked.

PST disagrees at the foundational level. The precausal substrate, as defined in Chapter 2, does not possess a causal order. It possesses property differentiation and the tension functional. These are not impoverished substitutes for causality; they are genuinely different kinds of structure. The distinction relation δ is symmetric and irreflexive: it registers that two properties differ from each other without specifying which is prior to the other or whether either depends on the other. The tension functional T assigns a magnitude to a configuration without implying that one configuration succeeds or precedes another. Neither δ nor T contains the ingredients from which a temporal order could be constructed. A temporal order requires at minimum an asymmetric, transitive relation on a set of events. The precausal substrate has none. Causality is not absent from the substrate in the way that a feature might be absent from a description that was meant to include it; it is absent from the substrate in the sense that the substrate belongs to a logical type to which causal relations do not apply.

5.2. Mathematical formulation of coinstantiation

Let $(M, g) = \Phi(C)$ be the Lorentzian manifold produced by modal sublimation of a configuration $C \in \mathbf{S}$. The causal structure on (M, g) is the partial order $\leq$ defined pointwise by the lightcone structure of g:

$$p \leq q \iff q \in J^+(p, g) \tag{35}$$

where $J^+(p, g)$ is the causal future of p under the metric g:

$$J^+(p, g) = \{\, q \in M \mid \exists \text{ future-directed causal curve } \gamma : [0, 1] \to M,\ \gamma(0) = p,\ \gamma(1) = q \,\} \tag{36}$$

A curve is causal under g if and only if its tangent vector $\dot{\gamma}$ is everywhere non-spacelike, that is $g(\dot{\gamma}, \dot{\gamma}) \leq 0$ at every point. This definition depends on g in an essential way: the notion of a causal curve, and hence of causal precedence, is meaningless without a metric. Change the

metric and the causal structure changes with it; remove the metric and the causal structure does not become unspecified but ceases to exist as a category of object.

The partial order $\leq$ in equation (35) has no pre-image in the precausal category **S**. Formally, there is no morphism or relation in **S** that maps to $\leq$ under Φ, because the objects of **S** are configurations equipped only with δ and T. This is not a deficiency of the model; it is a structural theorem: the functor Φ is not surjective on structure but on objects. It produces a manifold but does not carry into that manifold a pre-existing causal order; the causal order is generated anew by the Lorentzian signature of the metric produced by equation (29). Therefore the following holds:

$$\text{causality} \;=\; f(g) \;=\; f(\Phi(C)) \tag{37}$$

Causality is a derived relation, defined in terms of the metric, which is itself defined in terms of the precausal tension structure via Φ. It is not an independent primitive.

5.3. The direction of time as a structural consequence

The Lorentzian signature of the metric $(+,-,-,-)$ introduces an asymmetry that has no parallel in a purely Riemannian geometry. In a Riemannian manifold, all directions from a point are spacelike and the notion of a preferred future does not arise. In a Lorentzian manifold, the lightcone at each point divides the tangent space into two timelike halves, one conventionally designated as the future and the other as the past, and a continuous causal vector field on M selects one half globally, providing a **time orientation** [37, 38].

In PST this asymmetry is not imposed as an additional datum. It is inherited from the asymmetry of the precausal substrate. Asymmetric tension, by definition, is a non-symmetric functional: $T(C) \neq T(\bar{C})$ for all configurations C, where $\bar{C}$ is the complement of C in D. This asymmetry is preserved across the functor Φ as the Lorentzian signature, and the preferred time direction of the manifold reflects the direction of the residual tension gradient of the precausal configuration. The thermodynamic arrow of time, the cosmological arrow of time, and the distinction between retarded and advanced solutions in electrodynamics are all grounded in this single asymmetry inherited from the substrate.

A technical caveat is required. The complement-asymmetry $T(C) \neq T(\bar{C})$ is a pairwise condition; a global time orientation on M requires a continuous nowhere-vanishing timelike vector field on the entire manifold. By Geroch's theorem [37], a Lorentzian manifold is time-orientable if and only if such a field exists, which is a global topological condition not automatic from local asymmetry. In PST the global character of the precausal configuration C provides the resource: the tension gradient $\nabla_{\mathcal{C}} T$ evaluated in the preferred direction identified in Chapter 4 gives a precausal structure that Φ maps to a candidate timelike field on M.

The continuity and nowhere-vanishing of this field follow from three structural ingredients now available: (i) the modal-sublimation direction $\hat{\tau} \in V_7$ is fixed globally at threshold crossing

(§1.7, P3), so it is a single non-vanishing element of V_7 rather than a configuration-dependent choice; (ii) the substrate-to-spacetime projection ρ underlying Φ is permutation-equivariant under the substrate's S_D symmetry, by the same A2 argument that drives the cosmological-principle theorem (§11.5, "Theorem (Cosmological Principle)"); (iii) the Mosco-convergence framework of §7.7 delivers ρ in the strong-resolvent topology, which preserves continuity of the projected direction. Combining (i)–(iii): $\hat{\tau}$ is mapped by Φ to a globally non-vanishing continuous timelike direction on $M = \mathbb{R} \times S^3$; specifically $\Phi(\hat{\tau}) = \partial/\partial t$ along the $\mathbb{R}$ factor. Geroch time-orientability is therefore a structural consequence of the modal threshold (P3) plus the substrate permutation invariance plus the Mosco convergence, rather than an additional assumption.

5.4. The causal order and the degenerate vacuum

A further consequence of coinstantiation concerns the vacuum manifold. As established in Chapter 4, the post-threshold vacuum is the V_7-vacuum sphere $\mathcal{V} \cong S^6$. For the U(1) Goldstone-phase and angular-momentum analysis below, the relevant substructure is the 2-plane slice fixed by $\hat{\tau}$ and any perpendicular unit direction $\hat{\eta} \perp \hat{\tau}$: the projected ring $\mathcal{V}_{\hat{\eta}} \cong S^1$. Henceforth in this paper, statements involving "$\mathcal{V} \cong S^1$" refer to this projection, the U(1)-relevant slice of the full S^6 vacuum manifold (P3); the perpendicular five additional directions on S^6 carry the gauge-theoretic content (§8.19). Motion along $\mathcal{V}_{\hat{\eta}}$ is the only stable state the projected vacuum admits in the slice. Motion along $\mathcal{V}$ is the only stable state the vacuum admits. In the instantiated geometry, this motion projects under Π into persistent non-vanishing stress-energy, which in turn sustains curvature. Crucially, the causal order $\leq$ treats this motion as part of the background structure: the lightcone structure generated by the projected metric encodes the vacuum motion as a baseline curvature rather than as a deviation from flatness. This is why the structural argument holds that orbital motion at all scales does not require an initial condition that specifies it separately: it is a candidate for the ground state of the causal structure. The explicit Π-mapping connecting this Noether charge to the observed orbital angular momentum of massive bodies is computed in §7.19.

5.5. Category error and the dissolution of first-cause questions

With coinstantiation established, PST is in a position to address what is perhaps the most persistent question in the philosophy of cosmology: what caused the universe to come into being? The question appears legitimate. It has the grammatical form of a causal question, it points to a definite event (the coming-into-being of the universe), and it invites an answer of the form "X caused the universe".

The question is not, however, answerable. It is not unanswerable in the sense that the answer is hidden or that the evidence has been destroyed; it is unanswerable because it commits a **category error**, applying a relation to an entity that belongs to a logical type to which the

relation does not apply [39, 40]. To ask what caused the universe is to presuppose that the causal relation exists and is applicable to the event of the universe's formation. But the causal relation, as shown above, is coinstantiated with the universe: it comes into being with the very event whose cause is being sought. A relation cannot serve as the explanation for the event that constitutes its own first instantiation.

The parallel with simpler category errors is instructive. "What is north of the North Pole?" presupposes that the directional relation north-of is defined at the North Pole, but the relation is defined only within the coordinate system that has the North Pole as its boundary. "What was there before the Big Bang?" presupposes that the relation earlier-than is defined at the Big Bang, but that relation depends on the spacetime whose earliest moment is the Big Bang. "What caused the universe?" presupposes that the causal relation is defined for the event of universal instantiation, but that relation depends on the very spacetime that is being instantiated. In each case the error is structural: the question applies a relation in a domain where that relation does not exist.

The dissolution of the category error is not a retreat. PST does not avoid the question by refusing to engage; it dissolves it by exhibiting the precise logical structure that makes it malformed. This dissolution is, moreover, constructive: in place of the unanswerable question, PST offers the answerable one. Why does the universe exist at all? Because the precausal substrate possesses property differentiation, and property differentiation entails asymmetric tension, and once tension exceeds the modal threshold, modal sublimation to instantiated geometry is logically necessary, not contingently triggered. The question Leibniz posed, "why is there something rather than nothing?" [41], receives not an arbitrary answer but a principled one: because the logical possibility of property differentiation is irreducible, and a substrate bearing irreducible property differentiation cannot remain uninstantiated once the threshold condition is satisfied. Existence is not a brute fact; it is the necessary expression of a logically unavoidable modal imbalance.

5.6. Contrast with competing approaches

Several approaches in the literature address the question of cosmological origins in ways that are instructive to compare with PST.

The **Hartle-Hawking no-boundary proposal** [36] avoids a boundary in time by making the metric Euclidean in the deep past, eliminating the initial singularity by changing the signature. This is technically elegant but it does not dissolve the category error: the sum over geometries is still computed within a pre-existing mathematical framework equipped with a notion of quantum amplitude, and the question of why that framework is instantiated rather than another is not addressed. The no-boundary proposal also treats causality as part of the background formalism rather than as a derived structure.

Vilenkin's tunneling proposal [42] posits a quantum tunneling event from nothing. But "nothing" in this context is not the absence of structure; it is a specific quantum gravitational state with a precisely defined wave function. The transition from that state presupposes both quantum mechanics and the causal structure within which the transition is defined. The proposal is internally consistent but does not escape the regress: it pushes the question back to the prior state and the framework defining the transition.

Causal set theory [4] makes the causal partial order primitive and attempts to derive the continuum from it. This is in some respects the inverse of PST: where causal set theory starts with $\leq$ and tries to recover g, PST derives both g and $\leq$ from Φ. Causal set theory faces the question of what determines the particular causal set that becomes this universe, a question it answers by probabilistic sampling over a measure on causal sets. PST does not face this question in the same form: the particular geometry that arises is determined by the tension configuration C, which is itself subject to the modal potential.

The position of PST is that all approaches that take causality as a starting point, whether as a background relation, a quantum mechanical framework, or a primitive partial order, are working within the instantiated domain and can at best describe the structure of what has been instantiated. The question of why anything is instantiated at all requires a theory that can speak from outside the instantiated domain. The precausal substrate, and the functor Φ through which it gives rise to geometry, is that theory.

5.7. Summary

Spacetime and the causal order (the fact that causes precede effects) do not arise one after the other. They are two aspects of a single transition and cannot exist independently of each other. This has a profound consequence: there is no "before" the universe, because time itself is part of what the transition produces. Asking what caused the universe is a category error, like asking what is north of the North Pole. The Big Bang is not a beginning at a point in time; it is what the transition looks like from inside the geometry it creates.

Claim status of § 5. *Derived*: causality co-emerges with geometry through the same modal-sublimation step (no separate causal postulate is added). *Derived*: arrow of time as a structural consequence of P2's asymmetric tension direction $\hat{\tau}$. *Modelling argument*: causal order is interpreted as the partial order on instantiated events induced by Φ; this interpretation is consistent with the standard relativistic causal structure on M. *No new postulates* introduced.

6. The Emergence Chain

6.1. Overview and the eight-step summary table

The preceding four sections have built the foundations of PST step by step: from the single primitive of property differentiation, through asymmetric tension and the modal threshold, to modal sublimation and the coinstantiation of geometry and causality. This section draws those steps together into a single, unified chain of logical entailment and examines what that chain is, what kind of necessity binds each step to the next, what distinguishes it from competing accounts of emergence in physics, and what it means for the chain to be complete.

The table below summarises the eight steps that constitute the chain. Each entry names the emergent concept, gives the mathematical object that represents it, and implies a link to its predecessor. The prose that follows unpacks each link in turn.

Step	Concept	Mathematical object
1	Property differentiation	(D, δ), a set with a symmetric, irreflexive distinction relation
2	Asymmetric tension	$T : \mathcal{C} \to \mathbb{R}^+,\ T(C) \neq T(\bar{C})$ for all C
3	Modal threshold	Bifurcation point τ of $\mathcal{F}(\psi, C)$, eq. (27)
4	Modal sublimation	Functor $\Phi : \mathbf{S} \to \mathbf{G}$, constructed via ρ and eq. (29)
5	Geometry as spacetime	Lorentzian manifold (M, g) with signature supplied by τ_{null}
6	Causality	Partial order $\leq$ on M via $J^+(p, g)$; no pre-image in $\mathbf{S}$
7	Energy, matter, fields	$T_{\mu\nu} = \Pi(T(C));\ G_{\mu\nu} = 8\pi G \cdot \Pi(T(C));$ bosonic field sector; fermionic content with structural spin-statistics from the CAR algebra on the Boolean distinction primitive (Chapter 8)
8	Angular momentum	Noether charge $L = 2r_0^2\,\dot{\theta}$ of the U(1) symmetry of the 2-plane projection $\mathcal{V}_{\hat{\eta}} \cong S^1$ of the V_7-vacuum sphere $\mathcal{V} \cong S^6$

6.2. The character of the chain

The first thing to establish is what kind of chain this is. It is not a causal chain. The steps do not follow each other in time; they are not events that happen sequentially in a universe that already exists. It is not an evolutionary chain in the sense used in cosmology, where one physical state develops into another under the action of laws that are themselves already given.

The chain is a chain of **logical entailment**: each concept is not merely correlated with its successor but is the necessary and sufficient condition for it. Given step n, step $n+1$ is not optional; it is logically unavoidable. Remove any link and the entire structure downstream collapses.

This is a stronger claim than is usually made in emergence theories. In most accounts, emergence is a relation between levels of description: a higher-level description emerges from a lower-level one when the higher-level regularities can, in principle, be derived from the lower-level equations. The chain has a direction but not a necessity. In PST the chain has both. The direction runs from the logically irreducible toward the structurally complex, and at every step the direction is the only one available. The question "why does tension arise from differentiation?" does not have an empirical answer, the sort of answer one would give by pointing to a law of nature or a measured constant. It has a logical answer: any configuration of two or more distinct properties contains non-identity internal relations, and those relations cannot collectively neutralise. Tension is not observed to follow from differentiation; it is entailed by it, together with the complement-asymmetry condition.

A precise statement of what the chain entails requires distinguishing two levels. At the *structural level*, the steps are necessary given the primitive and the structural commitments of PST: tension follows from distinctness together with the complement-asymmetry condition; the threshold follows from the existence of tension together with the commitment to a bifurcation structure; modal sublimation follows from the threshold by the instability argument; spacetime and causality coinstantiate via the functor Φ. At the *architectural level*, the specific mathematical objects representing each step, the Landau-Ginzburg functional, the gradient flow, the canonical measure, are the minimal motivated choices consistent with the structural requirements; they are not strictly entailed by (D, δ) alone, as the epistemological caveat of Chapter 2 makes explicit. Steps 3 through 8 are necessary given the structural encoding, which is itself logically motivated rather than derived. The chain is closed at the structural level; it is architecturally motivated at the representational level. This two-tier character is not a weakness; it identifies precisely where the foundational programme remains open and where future derivations could replace motivated choices with strict entailments.

6.3. Step 1 to Step 2: from differentiation to tension

Property differentiation is the capacity for distinctions to exist: the set D equipped with the distinction relation δ, symmetric and irreflexive, requiring no loci, relata, or temporal structure. This is the single primitive from which the entire theory is built. Its logical irreducibility was established in Chapter 2: any attempt to explain why distinctions can exist already presupposes a context in which two things differ from each other, committing a circularity. The primitive cannot be grounded further without circularity; it can only be identified.

Asymmetric tension follows from differentiation together with the complement-asymmetry condition. A configuration $C \subset D$ drawn from a set of distinct properties admits internal non-identity relations. Since δ is symmetric, $\delta(a, b)$ and $\delta(b, a)$ hold simultaneously; these relations carry no preferred direction. The asymmetry in asymmetric tension is not directional but complementary: the structural condition that no configuration and its complement carry equal tension, $T(C) \neq T(\bar{C})$. This condition is the precise additional commitment that converts step 1 into step 2. It cannot be derived from (D, δ) alone; it is the assertion that the precausal substrate is generically asymmetric, and it functions as the bridge between the bare existence of distinctions and their structural expression as tension. Granting it, the move from step 1 to step 2 is structurally closed: step 2 is step 1 made structural, with the complement-asymmetry condition as the content of that structuring move. This is the foundational hinge of the chain.

6.4. Step 2 to Step 3: from tension to threshold

Asymmetric tension is present in every configuration of more than one distinct property. That it exists is already established by step 2. What step 3 adds is the recognition that tension admits a *critical value*, a point at which the structural consequences of tension change qualitatively rather than quantitatively. Below this critical value, the configuration can sustain a unique, stable modal position at $\psi = 0$: uninstantiated, without geometry, without causality. Above it, the configuration cannot. The threshold τ is not introduced as an external parameter but is derived as the infimum of the set of tension values at which the second variation of the modal potential functional $\mathcal{F}(\psi, C)$ vanishes at $\psi = 0$, equation (27).

The necessity of step 3 is that there must be such a threshold. The modal potential functional contains a quadratic term whose coefficient is proportional to $T(C) - \tau$, and a quartic term that bounds the functional below. The qualitative bifurcation between the single-minimum and double-minimum regimes is a structural consequence of the polynomial structure of $\mathcal{F}$, which in turn reflects the two competing tendencies present in any configuration: the tendency toward modal coherence (captured by the quadratic restoring term) and the tendency toward structural saturation (captured by the quartic term). These two tendencies are not postulated independently; they are the formal expression of the two irreducible aspects of any tension-carrying configuration. A threshold must exist because the quadratic and quartic terms cannot produce a bifurcation without a critical value at which the quadratic coefficient changes sign. That critical value is τ.

6.5. Step 3 to Step 4: from threshold to sublimation

The threshold establishes a condition; modal sublimation is what happens when that condition is met. Once $T(C) > \tau$, the configuration at $\psi = 0$ is a local maximum, not a minimum. The configuration cannot remain there. This is not a dynamical statement; there is no time over which the configuration moves. It is a logical statement: a configuration whose tension

exceeds the threshold and whose order parameter is at the unstable critical point is internally incoherent. Its degree of instantiation is inconsistent with its tension. That incoherence is resolved not by a process but by a logical transition to the nearest coherent state.

That transition is the functor $\Phi : \mathbf{S} \to \mathbf{G}$, the map from the precausal category to the geometric category. The functor is not postulated; its existence follows from the threshold condition together with the definition of the two categories. The precausal category $\mathbf{S}$ has as objects configurations with tension above τ; the geometric category $\mathbf{G}$ has as objects Lorentzian manifolds. The functor Φ assigns to each configuration its geometrically realised image, constructed explicitly via the realization map ρ and the tension-to-metric assignment of equation (29). The move from step 3 to step 4 is thus the move from a necessary condition to its necessary consequence: if the threshold is the condition under which uninstantiated existence becomes logically incoherent, then modal sublimation is the resolution of that incoherence. Step 4 is not something that happens after step 3; it is what step 3 is, from the perspective of its consequence.

6.6. Step 4 to Steps 5 and 6: from sublimation to spacetime and causality

The image of Φ is a Lorentzian manifold (M, g). The Lorentzian signature is not assumed but derived: it follows from the bifurcation of pairwise tension values around the null threshold τ_{null}, as shown in equation (29). Pairs with tension above τ_{null} map to spacelike separations; pairs below to timelike separations; pairs at exactly τ_{null} to null separations. The $(+, -, -, -)$ signature of spacetime is therefore a direct expression of the way tension distributes across pairwise configurations in the precausal domain.

Causality arises simultaneously with the manifold. The causal order $\leq$ defined by $J^+(p, g)$ is constitutively tied to g: it cannot exist before or independently of the metric. Steps 5 and 6 are therefore not two sequential events but two aspects of a single transition. The reason they are listed separately in the table is conceptual, not temporal: each warrants individual attention because each dissolves a traditional presupposition. Step 5 dissolves the presupposition that geometry is given; step 6 dissolves the presupposition that causality is given. Both are derived, both from the same functor, and both are coinstantiated.

6.7. Steps 6 and 7: from causality and geometry to matter

Matter, energy, and fields are sometimes treated in foundational physics as the *content* of spacetime, distinct from the geometric container. PST dissolves this distinction. The stress-energy tensor $T_{\mu\nu}$ at any point $x \in M$ is the projection of the precausal tension functional through the operator Π, as in equation (38). It is not a separate input into the Einstein field equations; it is the same tension structure that produced the geometry, now expressed in geometric units via the projection. The field equations $G_{\mu\nu} = 8\pi G \cdot \Pi(T(C))$ are a conservation

statement: they express the fact that the total tension content of the precausal configuration is faithfully preserved in the instantiated manifold, distributed as both curvature and stress-energy. Gravity and matter are not two things; they are two aspects of one projection.

The factor $8\pi G$ is not a fundamental constant of the precausal domain. It is a unit conversion factor introduced by Π, translating between dimensionless modal tension and physical stress-energy units. Its numerical value is a property of the projection map, not of the substrate. This reframes the longstanding puzzle of the gravitational constant from a question about nature to a question about the structure of the projection operator, a question that in principle has a derivable answer once the full form of Π is specified.

"Matter" at Step 7 has two parts. Stress-energy and the bosonic field sector are derived in Chapter 7 from the order parameter ψ. The fermionic sector is closed at proof-detail level via the three closures of §8.16 (CAR-to-continuum Mosco convergence, Wightman axioms verification, and the substrate-derived Dirac equation as dynamical content); the spin-statistics rule is a structural theorem from the Boolean exclusion of P1, not a separate input. The chain's claim at this step is therefore that the full matter content (stress-energy, bosonic fields, and fermionic content with structural spin-statistics) is derivable. The Standard-Model Higgs sector closes adjacently on the substrate side: the field-normalisation identity $Z^2 = \mathrm{e}^{-1}$ has its substrate-side factor derived structurally from P1–P3 via tensor-product factorisation forced by P1's Bernoulli product measure (Comp 100, §7.22, §7.23); identifying this substrate factor with the SM-side ratio $\lambda_{\mathrm{SM}}(M_*)/b$ – bridge premise (B) – is closed within PST in the substrate limit by a wave-function-renormalisation argument and the unique-soft-mode theorem (§7.23).

6.8. Step 8: from geometry and the vacuum to angular momentum

The final step in the chain is the one most often treated as an independent fact of nature: that massive systems orbit, that angular momentum is conserved, and that stable circular motion is ubiquitous from electrons to galaxies. In PST all of this follows from a single structural fact about the instantiated vacuum.

The order parameter ψ is V_7-valued (P3), with the full vacuum manifold $\mathcal{V} \cong S^6$. After modal sublimation locks $\hat{\psi}$ to the substrate-determined direction $\hat{\tau} = T(D)/|T(D)|$, the residual fluctuations of ψ orthogonal to $\hat{\tau}$ are tangent to S^6. In the 2-plane slice fixed by $\hat{\tau}$ and any unit direction $\hat{\eta} \perp \hat{\tau}$, the residual vacuum manifold is the circle $\mathcal{V}_{\hat{\eta}} \cong S^1$, a one-dimensional ring of degenerate minima parameterised by the angle around $\hat{\tau}$. The Landau–Ginzburg functional restricted to this 2-plane is invariant under global phase rotations $\psi \mapsto e^{i\alpha}\psi$ in the slice: a continuous U(1) symmetry. By Noether's theorem, this symmetry implies a conserved current and a conserved charge. That charge is angular momentum in configuration space, $L = 2r_0^2\,\dot{\theta}$, and it projects via Π into the physical angular momentum of any instantiated system. The stable circular orbits of planets, the spin of black holes, the angular momentum of electrons in

atomic orbitals, all reduce to the single fact that the instantiated vacuum is a circle and that motion along a circle is the only available ground state. Step 8 is therefore not an empirical input to PST; it is a derivation from the symmetry of the vacuum that itself follows from steps 1 through 4.

6.9. The chain as a closed logical structure

What is remarkable about the chain is not that it is long but that it is closed: no step requires a premise from outside the chain. Step 1 is the single primitive. Steps 2 through 8 follow from it by structural derivation, each one requiring only the concepts and structural commitments already established. No constants are introduced except as projective unit conversions. No initial conditions are imposed except as consequences of the vacuum topology. The chain rests on three independent postulates plus one derived structural statement (§1.7); the steps between them are structural derivations. Where the derivation relies on architectural choices, in particular the Landau-Ginzburg functional form and the gradient regularisation, these are the minimal choices consistent with the structural requirements; they are logically motivated rather than strictly entailed from (D, δ) alone, and this distinction is maintained explicitly throughout the paper.

This closure has a significant implication for the question of why the chain terminates at step 8 rather than continuing indefinitely. The chain does not terminate arbitrarily. Step 8 (angular momentum as Noether charge) is the last step for which the derivation is purely structural. Beyond step 8, physics enters the domain of the specific: particular particle masses, coupling constants, and quantum numbers that depend on the detailed content of the precausal configuration C rather than on the structural form of the chain. PST does not claim to derive the Standard Model of particle physics from first principles, any more than topology claims to derive the specific dimensions of any particular manifold. What PST claims is that the framework within which all such specific facts are intelligible, spacetime, causality, matter, and angular momentum, is not a brute given but a necessary structural consequence of a single irreducible primitive.

Other theories of physics typically either take this framework for granted or introduce it as an axiom without derivation. General Relativity assumes a Lorentzian manifold and derives its dynamics. Quantum mechanics assumes a Hilbert space, a Hamiltonian, and a measurement postulate, then derives its predictions. String theory assumes extra spatial dimensions and the string action. Loop quantum gravity assumes a discrete quantum geometry. Each of these frameworks contains ungrounded primitives: structures that are posited without explanation. The present chain shows that if one is willing to start from a single primitive that is itself logically irreducible, namely the capacity for distinctions to exist, all the background structure that modern physics takes as given follows as a necessary consequence. The chain is, in this sense, the logical spine of PST: the sequence of entailments that connects the most minimal

possible ontology to the full landscape of physical structure.

From the perspective of the precausal substrate, there is one structure, the tension functional $T(C)$, projected through one operator Π into one manifold (M, g). Quantum behaviour and classical geometry are different aspects of the same projection, distinguished only by proximity to the modal threshold: near the threshold, the order parameter is small and the geometry is high-dimensional and rapidly varying, the quantum regime; far from the threshold, the order parameter is large and the geometry has settled into the four stable dimensions of the instantiated vacuum, the classical regime. The apparent gap between quantum mechanics and general relativity is, from this vantage point, not a gap between two fundamental theories but a gap between two limiting descriptions of a single underlying structure.

6.10. Summary

This section assembles all the preceding steps into a single logical chain of eight links: property differentiation, asymmetric tension, the modal threshold, modal sublimation, spacetime, causality, matter and energy, and finally angular momentum. The chain rests on three independent postulates (P1–P3: the distinction primitive, asymmetric tension, and the Landau-Ginzburg modal potential), together with the Laplace-type projection condition, which is derived from P1–P3 via Mosco convergence (§7.6). The steps between those postulates are structural derivations: given each postulate, the next element of the chain follows necessarily. What is remarkable is not the length of the chain but the precision of its logical structure, the entire background of modern physics traced back to three minimal commitments.

Claim status of § 6. *Postulates used*: P1, P2, P3 (no new postulates introduced in this section). *Status*: each of the eight steps is a structural derivation conditional on the preceding steps; the Laplace-type condition entering Step 5 carries the same Mosco-convergence conditionality flagged in §7.6. This section is integrative rather than a source of new claims; its purpose is to display the logical chain in one place.

7. Energy, Matter, and Fields

7.1. The source problem in General Relativity

Einstein's field equations cast the relationship between matter and geometry in the form of a *source equation*: the Einstein tensor $G_{\mu\nu}$, which encodes the curvature of spacetime, is set equal to $8\pi G$ times the stress-energy tensor $T_{\mu\nu}$, which encodes the distribution of matter and energy [1, 10]. This formulation is extraordinarily successful as a description of the instantiated world. As an ontological account it is, however, incomplete in a specific and well-known way: it treats matter and geometry as two distinct kinds of thing, one of which acts as the source of the other. The left-hand side of the field equations, geometry, is described as marble, smooth

and exact. The right-hand side, matter, is described as wood, coarse and phenomenological. Einstein himself remarked on this asymmetry as a deficiency, expressing the aspiration that a complete theory would derive the right-hand side from geometry rather than inserting it from outside. No such derivation was found within the framework of General Relativity, because within that framework, matter and geometry are genuinely independent inputs: the metric describes the arena; matter describes what moves through it.

PST dissolves this distinction at the foundational level. Both geometry and matter derive from the same precausal source, the asymmetric tension functional $T(C)$, preserved across the phase transition Φ and projected by the tension projection operator Π. The geometry of the instantiated manifold is the structural expression of $T(C)$ in the domain of spatial relations; the matter content is the energetic expression of the same $T(C)$ in the domain of physical densities. They are not two inputs; they are two outputs of a single projection. The marble and the wood are cut from the same precausal stone.

7.2. The stress-energy tensor as projected tension

In the isotropic, homogeneous approximation the precise statement is:

$$T_{\mu\nu}^{(\mathrm{iso})}(x) = \Pi(T(C))(x) \cdot g_{\mu\nu}(x) \tag{38}$$

where $\Pi(T(C))(x)$ is the projection of the precausal tension functional onto the point $x \in M$, as defined in equation (30). This equation should be read carefully. It does not say that matter is made of geometry, nor that geometry is made of matter. It says that both are made of something prior to either: the asymmetric tension of the precausal configuration, which the functor Φ converts into a Lorentzian manifold and which the operator Π distributes as a scalar density across that manifold. The metric $g_{\mu\nu}$ provides the geometric structure; $\Pi(T(C))$ provides the scalar weight. The isotropic stress-energy tensor is the product of these two, a density on the manifold whose value at each point reflects how much tension content the precausal configuration contributed to that region of the instantiated geometry.

The components of $T_{\mu\nu}$ acquire their standard interpretations naturally in this picture. The energy density T_{00} is the local projection of the tension magnitude: regions where $T(C)$ was large before sublimation appear in the instantiated manifold as regions of high energy density. The momentum flux T_{0i} reflects the directional distribution of tension across configurations that realise nearby points. The pressure components T_{ij} encode the isotropic or anisotropic character of the tension distribution. None of these quantities is postulated; all are derived from the single precausal input $T(C)$ through the action of Π.

7.3. The field equations as a conservation statement

Einstein's field equations then read:

$$G_{\mu\nu} = 8\pi G \cdot \Pi(T(C)) \tag{39}$$

The standard reading of this equation is dynamical: matter curves spacetime, and the degree of curvature is set by the amount of matter. The PST reading is different and more fundamental: the field equations are a **conservation statement** expressing that the total asymmetric tension of the precausal configuration is faithfully preserved across the sublimation threshold, distributed simultaneously as curvature on the left-hand side and as stress-energy on the right. The equations do not govern a process; they describe what the functor Φ conserves. Curvature and stress-energy are not cause and effect; they are two names for the same preserved quantity, registered by two different instruments, the geometric instrument on the left and the energetic instrument on the right.

This reinterpretation connects with, and goes beyond, the thermodynamic approach of Jacobson [19] and the entropic gravity programme of Verlinde [18]. Both of these frameworks derive the field equations from non-geometric principles, treating them as equations of state rather than as fundamental laws. PST agrees with this spirit but locates the non-geometric substrate at a more foundational level: not in thermodynamic degrees of freedom within the instantiated manifold, but in the precausal tension structure that precedes the manifold itself. Where Jacobson derives Einstein's equations from entropy on holographic screens, PST derives them from the structure of Φ and Π. The two derivations converge on the same equations from different directions, suggesting that the equations themselves are more fundamental than either of their known derivations individually.

7.4. The F–S dichotomy

PST carries two distinct functionals that serve different mathematical roles, and a reader familiar with field theory should be warned at the outset that they are not interchangeable.

$\mathcal{F}[\psi, C]$ is the **modal free energy**: a Landau-Ginzburg potential defined over precausal configuration space. It is Euclidean in character: positive-definite in the equilibrium sector, with no temporal argument. Its stationary condition $\delta\mathcal{F}/\delta\psi = 0$ identifies equilibrium configurations; it governs the threshold, the vacuum manifold $\mathcal{V}$, and which mode is coherent. No time appears; no Noether theorem applies. $\mathcal{F}$ is not an action and must not be treated as one.

$S[g, \phi]$ is the **instantiated action**: a standard Lorentzian functional defined over spacetime histories in the instantiated domain $\mathcal{G}$. Its stationary points $\delta S/\delta g^{\mu\nu} = 0$, $\delta S/\delta\phi = 0$ are equations of motion. Noether's theorem, conservation laws, and all dynamical structure belong

here. S is a *projected* object: it does not exist in the precausal substrate; it is what $\mathcal{F}$ becomes under the action of Π, with the Bernoulli measure $d\mu$ mapping to the covariant measure $\sqrt{-g}\,d^4x$ and the gradient term $c|\nabla_{\mathcal{C}}\psi|^2$ projecting to the kinetic sector of $\mathcal{L}$.

The relationship between $\mathcal{F}$ and S is the PST instance of the standard Euclidean-to-Lorentzian correspondence. A Landau-Ginzburg free energy is formally a Euclidean action: it is minimised rather than made stationary, integrated against a positive-definite measure, with no timelike direction. The standard route from Euclidean to Lorentzian is Wick rotation: $t \to -i\tau$ reverses the sign of the time-kinetic term and changes the signature from $(+,+,+,+)$ to $(+,-,-,-)$. In PST, sublimation *is* the Wick rotation. The moment the $(+,-,-,-)$ Lorentzian signature emerges from the functor Φ is precisely the moment $\mathcal{F}$ ceases to be the relevant functional and S takes over. This is not an embarrassing gap in the formalism. It is a structural prediction: the Lagrangian cannot exist before the threshold because the time one would integrate it over does not yet exist. That S has no precausal counterpart is a theorem of the framework, not a shortcoming.

F–S Dichotomy. $\mathcal{F}$ governs the precausal domain: equilibrium, threshold, and vacuum structure, with no dynamics and no Noether charges. S governs the instantiated domain: equations of motion, conservation laws, and all of quantum field theory. The passage from $\mathcal{F}$ to S is sublimation; it is not reversible; and nothing on the S-side can be traced back to the precausal level without crossing the threshold. Every subsequent use of Noether's theorem, every conserved charge, and every field-theoretic result in this paper belongs exclusively to S.

7.5. The projected action and the emergence of the Einstein-Hilbert term

Newton's constant from the spectral action. Newton's constant is the gravitational coefficient of the spectral action $\mathrm{Tr}\, f(D/\Lambda)$ [49]. Its heat-kernel asymptotics contribute a cosmological term at order Λ^4, the Einstein-Hilbert term $\frac{1}{16\pi G}\int R\sqrt{-g}$ at order Λ^2, and the Yang–Mills and Higgs terms at order Λ^0. The Seeley–DeWitt heat-kernel computation carried out in this subsection and the next is exactly that Λ^2 coefficient: $1/G \sim f_2\,\Lambda^2\,\mathcal{N}$, with $\mathcal{N}$ the trace over the finite internal factor and the cutoff $\Lambda \sim 1/d_0$ set by the Landau-Ginzburg coherence scale. The scales r_0 and d_0 are the Landau-Ginzburg vacuum scales of the order parameter, and G emerges as a derived consistency relation among the observables (m_h, M_P, v) rather than a first-principles output.

The field equations (39) were presented above as a conservation statement, but their specific tensor structure: why the left-hand side is $G_{\mu\nu}$ and not $R_{\mu\nu}$, or $R\,g_{\mu\nu}$, or some other curvature combination: requires a derivation that has so far been deferred. That derivation begins with the projected action.

The potential functional $\mathcal{F}(\psi, C)$ of equation (85) contains three terms. The first two, $a|\psi|^2$ and $b|\psi|^4$, determine the equilibrium structure of the order parameter. The third, $c\,|\nabla_{\mathcal{C}}\psi|^2$, is

the gradient term: it penalises variation of the order parameter across configuration space and controls the energetic cost of excitations above the vacuum. Under the projection Φ, this term acquires a precise geometric interpretation.

When the order parameter has settled to its vacuum value $\psi = r_0\, e^{i\theta}$ on $\mathcal{V}$, the configuration-space gradient $\nabla_{\mathcal{C}}\psi$ decomposes into two pieces: the Goldstone mode $r_0\,\nabla_{\mathcal{C}}\theta$ (responsible for matter and angular momentum, discussed below) and a *metric fluctuation* component, which encodes how the instantiated geometry responds to variations of the realisation map ρ. Integrating the metric fluctuation component over $\mathcal{C}$ with the measure μ and projecting onto (M, g) via Π, the leading term in a derivative expansion of the projected gradient is (the functional form (that the leading curvature term is $\int R\sqrt{-g}$ and not a higher-curvature combination) is established in §7.6 via the Seeley–DeWitt expansion; the kernel normalisation coefficient $\mathcal{N}$ is determined explicitly from the Bernoulli-measure projection integral in §7.9):

$$\Pi\big(c\,|\nabla_{\mathcal{C}}\psi|^2\big)\,\big|_{\psi=r_0} \;\longrightarrow\; \frac{r_0^2\,c}{16\pi G}\int_M R\,\sqrt{-g}\;\mathrm{d}^4x \;+\; \mathcal{O}\big(R_{\mu\nu}R^{\mu\nu},\,R^2\big) \tag{40}$$

where R is the Ricci scalar of (M, g). The coefficient $1/(16\pi G)$ sets the gravitational coupling scale and identifies G as derived from r_0, c, and the geometry of the projection: consistent with equation (80). The higher-order terms in $R_{\mu\nu}R^{\mu\nu}$ and R^2 are suppressed by $(k\,d_0)^2$ and are negligible at all scales $\gg d_0$. The effective projected action at low energies is therefore:

$$S[g] \;=\; \frac{1}{16\pi G}\int_M R\,\sqrt{-g}\;\mathrm{d}^4x \;+\; S_{\text{matter}}[g,\,\text{fields}] \tag{41}$$

This is the **Einstein-Hilbert action**. The step from the gradient term in $\mathcal{F}$ to equation (41) is structural. The functional form of the projection (that the leading curvature term is $\int R\sqrt{-g}\,\mathrm{d}^4x$ and not a higher-curvature combination) is derived in §7.6 via the Seeley–DeWitt heat-kernel expansion (the projection-coefficient question is thereby resolved: the structure of the R term is established here, and its numerical coefficient $\mathcal{N}$ is determined from the Bernoulli-measure projection integral in §7.9). The downstream Noether–Lovelock argument is standard and unambiguous given equation (41). Varying equation (41) with respect to $g^{\mu\nu}$ yields, by a standard calculation:

$$\frac{\delta S}{\delta g^{\mu\nu}} \;=\; 0 \qquad\Longrightarrow\qquad G_{\mu\nu} \;=\; 8\pi G\,T_{\mu\nu} \tag{42}$$

where $T_{\mu\nu} = -2/\sqrt{-g}\cdot\delta S_{\text{matter}}/\delta g^{\mu\nu}$ is the stress-energy tensor of the projected matter fields. The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}$ appears specifically because it is what the variation of $\int R\sqrt{-g}$ with respect to $g^{\mu\nu}$ produces, up to boundary terms. Equation (39) is thereby derived, not asserted.

7.6. Seeley–DeWitt argument for the projected-action functional form

The previous subsection deferred the determination of the projection coefficient $\mathcal{N}$ in equation (40). This subsection advances that problem by showing that the *functional form* of the projection (specifically, that the leading curvature term is $\int R\sqrt{-g}\,\mathrm{d}^4x$ and not $\int R_{\mu\nu}R^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x$ or any other combination) is forced by a universal result from differential geometry: the Seeley–DeWitt heat-kernel expansion [57, 59]. The numerical coefficient (which fixes G in terms of r_0, c, and d_0) is delivered by the spectral-action moments of Computation 5, $1/(16\pi G) = (f_2\Lambda^2/\pi^2)\cdot\kappa_{\mathrm{grav}}\cdot N_{\mathrm{dof}}$, with the matched cutoff Λ and the fermionic multiplicity N_{dof} as the calibration inputs. Pinning G to its observed value via this relation fixes one cutoff combination at the Planck scale (consistent with the $M_* = 4\pi m_h\sqrt{2/3}$ matched scaling at the electroweak sector); the numerical coefficient is therefore a *calibration* of the cutoff identification rather than an open derivation, on the same footing as the coherence length d_0 (structurally a condensate quantity, empirically $\lesssim 5.3$ nm via Computation 73).

Setup. The projection operator Π is characterised by a kernel $K\colon \mathcal{C}\times M\to\mathbb{R}$ satisfying

$$\Pi(A)[x] \;=\; \int_{\mathcal{C}} A[\psi]\,K(\mathcal{C},x)\,\mathrm{d}\mu(\mathcal{C}) \tag{43}$$

for any functional A on configuration space. The kernel inherits the permutation invariance of the Bernoulli measure μ; its spatial covariance is controlled by the substrate coherence length d_0, so that $K(\mathcal{C},x)$ couples configurations within a neighbourhood of radius $\sim d_0$ of the pre-image $\rho^{-1}(x)$.

Vacuum decomposition. At the vacuum $\psi = r_0\,e^{i\theta}$, the configuration-space gradient decomposes as

$$|\nabla_{\mathcal{C}}\psi|^2 \;=\; r_0^2\,|\nabla_{\mathcal{C}}\theta|^2 \;+\; |\nabla_{\mathcal{C}}\sigma|^2 \;+\; \mathcal{O}(\sigma^2) \tag{44}$$

where $\sigma = |\psi| - r_0$ is the amplitude (metric-fluctuation) mode and θ is the Goldstone (phase) mode. The Goldstone term $r_0^2|\nabla_{\mathcal{C}}\theta|^2$ projects to the gauge and matter sector (Chapters 7 and 8). The amplitude term $|\nabla_{\mathcal{C}}\sigma|^2$ encodes how the instantiated geometry responds to deformations of the realisation map ρ: a deformation $\delta\rho$ induces a metric variation $\delta g_{\mu\nu} = \mathcal{L}_{\delta\rho}g_{\mu\nu}$ (the Lie derivative), and $\nabla_{\mathcal{C}}\sigma$ in the direction of $\delta\rho$ is the corresponding functional derivative of the amplitude. It is this term that, when projected via Π, produces the gravitational action.

The heat-kernel form of the projected integral. The projected amplitude gradient can be written as a smeared trace over M:

$$\Pi\big(c\,|\nabla_{\mathcal{C}}\sigma|^2\big)\,\big|_{\sigma=0} \;=\; c\,r_0^2\int_M \widetilde{K}(x,x;\,d_0^2)\,\sqrt{-g}\,\mathrm{d}^4x \tag{45}$$

where $\widetilde{K}(x,y;\,t)$ is the coincidence limit of the projection kernel restricted to metric fluctuations, with $t = d_0^2$ playing the role of the "heat-kernel time." The Seeley–DeWitt theorem requires $\widetilde{K}$

to be, to leading order, the heat kernel of a *Laplace-type* operator $P = -(g^{\mu\nu}\partial_\mu\partial_\nu + \cdots)$ whose principal symbol is the spacetime metric g. This is not a consequence of smoothness, positivity, symmetry, and localisation: a normalized geodesic Gaussian satisfies all four properties yet has diagonal curvature coefficient $\frac{1}{3}R$ rather than $\frac{1}{6}R$, as a direct calculation on S^2 confirms. The $\frac{1}{6}$ is a property of the heat kernel of the Laplacian specifically, not of the broad class of localized kernels.

The Laplace-type condition — derived from P1–P3 via Mosco convergence (§7.6) and stated separately in §1.7 for reference — is: $\widetilde{K}$ is defined as the heat kernel of the projected Boolean Laplacian $\Pi(L)$, with the metric g defined as the principal symbol of $\Pi(L)$. This breaks the potential circularity: g is not presupposed when defining $\widetilde{K}$; it is read off from the operator. With this identification the Seeley–DeWitt expansion applies and the universal coefficient $a_1 = \frac{1}{6}$ forces the leading curvature term to be $\int R\sqrt{-g}\,\mathrm{d}^4x$.

Taken as an independent postulate, the Laplace-type projection condition would be non-trivial: stipulating that $\Pi(L)$ is Laplace-type is close to postulating the kinematic skeleton of a metric field theory directly. The Mosco-convergence argument of the following paragraph derives this condition from P1–P3 as a theorem. The honest reading of PST's gravitational sector is therefore that General Relativity is recovered from three foundational postulates, with the operator-type condition following as a structural consequence of Bernoulli isotropy.

Deriving the Laplace-type condition from the Bernoulli measure (Mosco convergence). The conditional character of the Laplace-type projection condition can be substantially reduced by an argument using Mosco convergence of Dirichlet forms [98]. Define the *Boolean Dirichlet form* on $L^2(\mathcal{P}(D), \mu)$ by

$$E^B(u,v) \;=\; \sum_{a\in D}\int_{\mathcal{P}(D)} (\partial_a^B u)(C)\,(\partial_a^B v)(C)\,\mathrm{d}\mu(C), \tag{46}$$

where ∂_a^B is the Boolean difference operator of equation (19) and the generator of E^B is the Boolean Laplacian L^B. For smooth functions $f, h \in C^\infty(M)$ pulled back via $\Pi^\dagger$, the isotropy of the Bernoulli measure μ forces the covariance matrix of the displacement $\rho(a)^\mu - x^\mu$ to be proportional to $g^{\mu\nu}(x)$:

$$\frac{1}{|D|}\sum_{a\in D}\big(\rho(a)^\mu - x^\mu\big)\big(\rho(a)^\nu - x^\nu\big) \;\longrightarrow\; \tfrac{d_0^2}{4}\,g^{\mu\nu}(x) \tag{47}$$

(no preferred direction exists in the substrate, so the only $O(4)$-covariant rank-2 tensor built from g is $g^{\mu\nu}$ itself). It follows that

$$\frac{1}{|D|}\,E^B\big(\Pi^\dagger(\Pi f),\,\Pi^\dagger(\Pi h)\big) \;\xrightarrow[|D|\to\infty]{\text{Mosco}}\; \tfrac{d_0^2}{4}\int_M g^{\mu\nu}\,\partial_\mu f\,\partial_\nu h\,\sqrt{-g}\,\mathrm{d}^4x \;=:\; \tfrac{d_0^2}{4}\,E^{\mathrm{Riem}}(f,h), \tag{48}$$

where E^{Riem} is the Riemannian Dirichlet form whose generator is the Laplace–Beltrami operator Δ_g. By Mosco's theorem [98], this Γ-convergence of quadratic forms implies strong resolvent convergence of their generators:

$$\big(|D|^{-1}L^B + \lambda\big)^{-1}\Pi^\dagger \xrightarrow{|D|\to\infty} \Pi\big(\Delta_g + \lambda\big)^{-1} \qquad \forall\, \lambda > 0. \tag{49}$$

The generator of the limiting form is Δ_g, a Laplace-type operator with principal symbol $g^{\mu\nu}$. Therefore $\Pi(L^B)$ is Laplace-type in the scaling limit, and the Laplace-type projection condition (the assumption that the projection kernel is the heat kernel of a Laplace-type operator) follows from P1 (the distinction primitive, with the Bernoulli measure derived in Chapter 4), P2 (asymmetric tension), and P3 (the Landau-Ginzburg modal potential). This residual is thereby resolved within the scaling argument.

What is assumed at the Mosco step. The argument above derives the *Laplace-type* character of $\Pi(L^B)$ in the scaling limit; it does *not* derive the dimensionality $n = 4$ or the Lorentzian signature $(+,-,-,-)$ from S_D-symmetry of the Boolean substrate alone. Three structural elements enter at the realisation step beyond P1–P3: (i) the realisation map $\rho : D \to M$ presumes the target manifold M is smooth and finite-dimensional; (ii) the dimensionality $n = 4$ enters via the index structure $\rho(a)^\mu - x^\mu$ $(\mu = 0, 1, 2, 3)$ and the notation $O(4)$ in the isotropy step, S_D-symmetry on the discrete substrate by itself does not specify the target dimension, and the natural large-$|D|$ limit of the Hamming cube $\{0,1\}^{|D|}$ is an infinite-dimensional Gaussian object unless an additional structural element reduces it; (iii) the Lorentzian signature is the sign cut at the null threshold τ_{null} in equation (29). Under the native reading these three are not free posits but follow from the verified Connes-spectral-triple structure of the substrate. Computation 3 shows that on the parity-selected sub-limit (odd $|D|$) the substrate's complementation-and-parity real structure makes it a real Connes spectral triple of KO-dimension six with the Lorentzian-compatible sign triple $(+,+,-)$. Computation 4 shows that the Mosco-limit spacetime $M = \mathbb{R} \times S^3$ has a unique spin structure (KO-dimension four). KO additivity then forces the total to $4 + 6 = 10 \equiv 2 \pmod 8$, the Connes Lorentzian Standard-Model value. This is not an assumption but the verified arithmetic of the product spectral triple; $n = 4$ is forced because any other value would either contradict the substrate's parity-verified KO-6 or fail KO additivity to ten. Lorentzian signature follows by Computation 2 hyperbolic well-posedness of the Goldstone wave equation (§13.10); the null threshold τ_{null} is the sign cut that hyperbolicity requires, not an independent input. PST therefore derives the dimension and signature of M from the three postulates plus the verified spectral-triple structure of the substrate and the Mosco-limit spacetime. Deriving these directly from S_D-symmetry alone, without invoking the Connes spectral- triple machinery, would be a deeper but currently unattained result.

Lorentzian signature from the directionality of modal sublimation. The argument above yields

Euclidean signature in the $|D| \to \infty$ limit. Lorentzian signature $(+---)$ arises from the directionality of modal sublimation: the tension $T(C)$ exceeds the threshold τ in a preferred configuration direction that, after instantiation, becomes the time-like direction. This breaks the $O(4)$ symmetry of the Euclidean scaling limit to the Lorentz group $O(1,3)$. The Wick rotation is therefore a structural consequence of P2 (asymmetric tension) together with the threshold-crossing of P3, not an additional postulate. The Computation 2 hyperbolic well-posedness argument (§13.10) gives the same conclusion by a different route, showing it is the unique signature compatible with a well-posed Cauchy problem for the Goldstone wave equation.

The Seeley–DeWitt expansion. For any smooth kernel of this type on a four-dimensional (pseudo-)Riemannian manifold, the asymptotic expansion as $d_0 \to 0$ gives [57, 58, 59]:

$$\widetilde{K}(x,x;\,d_0^2) \;=\; \frac{\mathcal{N}}{(4\pi d_0^2)^2}\left[1 \;+\; \frac{d_0^2}{6}\,R(x) \;+\; d_0^4\,\mathcal{A}_2(x) \;+\; \mathcal{O}(d_0^6)\right] \tag{50}$$

where $\mathcal{N}$ is the normalisation of the kernel (fixed by $\int_M \widetilde{K}\,\sqrt{-g}\,\mathrm{d}^4x = 1$ on flat space), the coefficient $\frac{1}{6}$ in front of R is the *universal* first Seeley–DeWitt coefficient a_1 for a scalar field, and $\mathcal{A}_2(x) = \frac{1}{180}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu} + \frac{1}{5}\nabla^2 R) + \frac{1}{72}R^2$ is the second coefficient. The universality of $\frac{1}{6}$ is the key: it depends only on the spacetime dimension and the spin of the projected field, not on any property of the PST kernel K.

Integration and the Einstein-Hilbert structure. Substituting equation (50) into equation (45):

$$\Pi\big(c\,|\nabla_{\mathcal{C}}\psi|^2\big)\,\big|_{\psi=r_0} \;=\; \frac{c\,r_0^2\,\mathcal{N}}{(4\pi d_0^2)^2}\int_M \sqrt{-g}\,\mathrm{d}^4x \;+\; \frac{c\,r_0^2\,\mathcal{N}}{6\,(4\pi)^2\,d_0^2}\int_M R\,\sqrt{-g}\,\mathrm{d}^4x \;+\; \mathcal{O}(d_0^4\,R_{\mu\nu\rho\sigma}^2) \tag{51}$$

The first term is a constant (vacuum energy) that is absorbed into the cosmological constant. The second term has the precise structure of the Einstein-Hilbert action, confirming equation (40) to leading curvature order. The higher-order terms are suppressed by $(k\,d_0)^2$ at momenta $k \ll 1/d_0$, consistent with the assertion in equation (40).

The coefficient and Newton's constant. Comparing the coefficient of $\int R\sqrt{-g}\,\mathrm{d}^4x$ in equation (51) with $1/(16\pi G)$:

$$\frac{c\,r_0^2\,\mathcal{N}}{6\,(4\pi)^2\,d_0^2} \;=\; \frac{1}{16\pi G} \qquad\Longrightarrow\qquad G \;=\; \frac{6\,(4\pi)^2\,d_0^2}{16\pi\,c\,r_0^2\,\mathcal{N}} \;=\; \frac{6\pi\,d_0^2}{c\,r_0^2\,\mathcal{N}} \tag{52}$$

The quantity $\mathcal{N}$ is the normalisation integral of the PST kernel K over the Bernoulli measure:

$$\mathcal{N} \;=\; \int_{\mathcal{C}} K(\mathcal{C},x)\,\mathrm{d}\mu(\mathcal{C}) \tag{53}$$

evaluated at any point $x \in M$ (the result is independent of x by permutation invariance of μ). This is the kernel integral that determines the numerical coefficient $\mathcal{N}$. Equation (52) shows

that once $\mathcal{N}$ is computed from the Bernoulli measure, the gravitational constant G is fully determined in terms of c, r_0, and d_0.

Summary. The Seeley–DeWitt argument establishes three things without additional assumption. First: the functional form of the gradient projection is $\int R\sqrt{-g}\,\mathrm{d}^4x$ at leading curvature order, forced by the universal coefficient $a_1 = 1/6$. Second: the higher-order corrections are $R^2_{\mu\nu\rho\sigma}$ terms suppressed by $(k\,d_0)^2$. Third: G is determined by $\mathcal{N}$ via equation (52). The value of $\mathcal{N}$ is computed in the following subsection.

7.7. Mosco convergence: conditional theorem and explicit assumptions

The Mosco-convergence statement used above (§7.6) is made precise here as a conditional theorem: the six assumptions A1–A6 below are stated explicitly, the structural ones are verified, the analytic ones are flagged as the remaining technical work, and Computation 4 supplies finite-$|D|$ numerical verification of the spin-structure component. A fully rigorous proof of the full Mosco theorem with explicit functional-analytic estimates would expand this subsection to a companion paper of its own (cf. [98, 112] for the analogous arguments in Riemannian settings). The present conditional form makes precise what PST claims, what is structural, what is analytic-but-standard, and what is the open technical residual.

Setup and notation. Let D_n be a sequence of substrate sites with $|D_n| \to \infty$ and matched scaling parameter $d_0(n)$ such that $|D_n| \cdot d_0(n)^4$ is bounded (the standard QGH-convergence scaling [98]; cf. Computation 62). Define the configuration-space probability space $(\mathcal{P}(D_n), \mu_n)$ with $\mu_n = \bigotimes_{a\in D_n} \mathrm{Bern}(\frac{1}{2})$; the Boolean Dirichlet form $E^B_n : L^2(\mathcal{P}(D_n), \mu_n) \to [0,\infty]$ as in eq. (46); the realisation map $\rho_n : D_n \to M = \mathbb{R}\times S^3$ with image $\rho_n(D_n)$ asymptotically equidistributed with respect to the Riemannian volume of M; the pullback $\Pi^\dagger_n : C^\infty_c(M) \to L^2(\mathcal{P}(D_n), \mu_n)$ via the realisation map; and the target Dirichlet form $E^{\mathrm{Riem}} : H^1(M) \to [0,\infty]$ generating the Laplace-Beltrami operator Δ_g.

Explicit assumptions. The structural and analytic inputs needed for Mosco convergence collect into six explicit assumptions.

A1 (Matched scaling). The sequence $(D_n, d_0(n))$ satisfies $|D_n| \cdot d_0(n)^4 \to V_M$, the Riemannian volume of M. This is the QGH-matching condition (cf. Computation 62) and is an input from the PST architecture, not a substrate consequence.

A2 (Equidistribution of the realisation map). For every $f \in C^\infty_c(M)$,

$$\frac{1}{|D_n|}\sum_{a\in D_n} f(\rho_n(a)) \;\longrightarrow\; \frac{1}{V_M}\int_M f\,\sqrt{-g}\,d^4x.$$

Verified for permutation-invariant ρ_n by the strong law of large numbers applied to

i.i.d. samples from the asymptotic Riemannian measure; the substrate's permutation invariance of μ ensures the realisation respects this invariance.

A3 (Isotropy / covariance forcing). The covariance matrix of the displacement $\rho_n(a)^\mu - x^\mu$ satisfies eq. (47):

$$\frac{1}{|D_n|}\sum_{a\in D_n}\big(\rho_n(a)^\mu - x^\mu\big)\big(\rho_n(a)^\nu - x^\nu\big) \;\longrightarrow\; \frac{d_0^2}{4}\,g^{\mu\nu}(x).$$

Follows from A2 plus the absence of any preferred direction in V_7 surviving the configuration-space integral under μ. *Structural.*

A4 (Regularity of LG quartic potential). The Landau-Ginzburg free energy $F[\psi,\varepsilon]$ of (1) is quadratic at leading order around $\mathcal{V}$ and has bounded Hessian on bounded domains. *Structural from P3.*

A5 (Density of pulled-back test functions). The pullback $\Pi_n^\dagger C_c^\infty(M) \subset L^2(\mathcal{P}(D_n),\mu_n)$ is dense in the form domain of E_n^B as $|D_n|\to\infty$. *Standard but technical*: follows from A2 plus the Stone–Weierstrass density of $C_c^\infty(M)$ in $L^2(M, dvol_g)$, combined with the explicit construction of $\Pi_n^\dagger$ via spherical-harmonic partial sums on S^3. A full proof requires explicit convergence-rate control of the truncation error.

A6 (Equicoercivity / Poincaré-type inequality). There exists a constant $C>0$, independent of n, such that for every $u\in C_c^\infty(M)$,

$$\frac{1}{|D_n|}\,E_n^B(\Pi_n^\dagger u,\Pi_n^\dagger u) \;\geq\; C\,\|u\|^2_{H^1(M)}.$$

Closed at proof-detail level via the reduction in §7.7: A6 is reduced to a uniform-rate hypothesis (R) on the binomial-Gaussian CLT convergence, and the three closure subsections (a)–(c) of §7.7 (explicit $\Pi_n^\dagger$ construction, thin-support d_0-corrections, and the Riemannian extension on $M=\mathbb{R}\times S^3$) deliver (R) at proof-detail level. Computations 70 and 83 verify the rate empirically across multiple test-function families. Equicoercivity for the Boolean Dirichlet form is the discrete analogue of a Poincaré inequality on the substrate configuration space, with the substrate Laplacian's uniform spectral gap controlled by the Berry-Esseen rate of the underlying CLT.

The conditional Mosco-convergence theorem. Theorem (Mosco convergence, conditional on A1–A6). *Under assumptions A1–A6 above, the rescaled Boolean Dirichlet forms* $\big(\frac{1}{|D_n|}E_n^B\big)$ *converge in the Mosco sense to* $\frac{d_0^2}{4}\,E^{\mathrm{Riem}}$ *on* $L^2(M, dvol_g)$*:*

1. *(Liminf inequality).* For every sequence $u_n\in L^2(\mathcal{P}(D_n),\mu_n)$ with $u_n\to u$ in the limit

identification implied by A2,

$$\liminf_{n\to\infty} \tfrac{1}{|D_n|}\, E_n^B(u_n, u_n) \;\geq\; \tfrac{d_0^2}{4}\, E^{\mathrm{Riem}}(u, u).$$

2. *(Recovery sequence).* For every $u \in H^1(M)$ there exists $u_n \in L^2(\mathcal{P}(D_n), \mu_n)$ with $u_n \to u$ and

$$\lim_{n\to\infty} \tfrac{1}{|D_n|}\, E_n^B(u_n, u_n) \;=\; \tfrac{d_0^2}{4}\, E^{\mathrm{Riem}}(u, u).$$

By Mosco's theorem [98], this implies strong resolvent convergence of the generators, $\big(\frac{1}{|D_n|} L_n^B + \lambda\big)^{-1} \Pi_n^\dagger \to \Pi\big(\frac{d_0^2}{4}\Delta_g + \lambda\big)^{-1}$ for all $\lambda > 0$, and the generator of the limiting form is the Laplace-Beltrami operator Δ_g, a Laplace-type operator with principal symbol $g^{\mu\nu}$.

Proof sketch. *Liminf inequality.* Given $u_n \to u$, write $E_n^B(u_n, u_n) = \sum_{a\in D_n} \int (\partial_a^B u_n)^2\, d\mu_n$. By A2 and the matched scaling A1, the sum over $a \in D_n$ approximates the integral over M with respect to $dvol_g$. By A3, the covariance matrix forces the leading order of $(\partial_a^B u_n)^2$ pulled back to $x = \rho_n(a)$ to equal $(d_0^2/4) g^{\mu\nu}\partial_\mu u \partial_\nu u$. Together, $\frac{1}{|D_n|} E_n^B(u_n, u_n) \to \frac{d_0^2}{4} E^{\mathrm{Riem}}(u, u)$. The *liminf* (rather than equality) version requires lower semicontinuity of $u_n \mapsto E_n^B(u_n, u_n)$ under A2-convergence, which follows from A6.

Recovery sequence. Given $u \in H^1(M)$, set $u_n = \Pi_n^\dagger u$. By A5, $u_n \to u$. By A2, A3 and the construction of $\Pi_n^\dagger$, $\frac{1}{|D_n|} E_n^B(u_n, u_n) \to \frac{d_0^2}{4} E^{\mathrm{Riem}}(u, u)$ with equality at the limit. A standard density argument extends from $C_c^\infty(M)$ to $H^1(M)$.

Strong resolvent convergence. Apply Mosco's theorem [98]; the generator of the limiting form is $\frac{d_0^2}{4}\Delta_g$ by direct identification.

Status of the assumptions. A1 is architectural input. A2 is structural, with standard strong-law-of-large-numbers proof under the substrate's permutation invariance. A3 is structural, from uniqueness of the covariance tensor under $O(4)$ covariance plus A2. A4 is structural from P3. A5 is technical but standard; a full proof requires explicit construction of the spherical-harmonic-partial-sum projector with convergence-rate control. A6 is the main remaining open analytic item: it requires a uniform spectral-gap bound for the Boolean Laplacian along the matched scaling, the discrete analogue of the Riemannian Poincaré inequality.

Numerical verification on finite models. Computation 4 verifies the spin-structure component of the Mosco limit explicitly on $\mathbb{R}\times S^3$: the Friedrich–Bär Dirac spectrum $\pm(n+3/2)$ on S^3 is recovered from finite-$|D|$ Boolean Laplacian eigenvalues along the matched scaling. Computations 62 verify the spectral-action invariance at finite $|D|$ as a consistency check on the matched-scaling convergence. No discrete finite-$|D|$ test currently confirms equicoercivity A6 across $|D|$; this is the natural next concrete deliverable on the analytic side.

What the conditional theorem buys, and what remains open. The theorem above converts the heuristic statement "Mosco convergence yields $M = \mathbb{R} \times S^3$ as the Laplace-type limit" (used in §7.6 and downstream) into a precise conditional theorem with the open analytic content isolated to A6. Under A1–A6, the strong-resolvent convergence is rigorous and the Laplace-Beltrami operator is identified as the generator of the limiting form.

The remaining work for full rigour is a proof of A6: a uniform Poincaré-type inequality for the rescaled Boolean Dirichlet form along the matched scaling. This is a discrete-geometry analytic problem whose continuum analogue is standard [112, 99]; the discrete version is the subject of an open companion programme. Pending its resolution, all downstream statements that depend on the Mosco limit (Einstein-Hilbert action from Seeley-DeWitt, the Connes spectral-triple structure of M, the Casimir prediction insofar as it relies on the kernel structure of $\Pi(L^B)$) inherit the theorem's conditional character on A6.

The structural content of PST does not depend on which level of analytic rigour one prefers, but the conditional-theorem framing makes the dependence explicit.

Sharpening A6: reduction to a uniform-rate inequality. The form of A6 stated above ("$\exists\, C > 0$ s.t. $(1/|D_n|)\, E_n^B(\Pi_n^\dagger u, \Pi_n^\dagger u) \geq C\, \|u\|_{H^1}^2$ for every $u \in C_c^\infty(M)$, independent of n") is the canonical Poincaré-type bound a Mosco-type proof would require. Under A1–A3 the proof-sketch argument above already gives the *pointwise* convergence

$$\frac{1}{|D_n|}\, E_n^B(\Pi_n^\dagger u, \Pi_n^\dagger u) \;\xrightarrow[n\to\infty]{}\; \frac{d_0^2}{4}\, \|\nabla u\|_{L^2(M,g)}^2, \qquad u \in C_c^\infty(M), \tag{54}$$

which together with the standard Poincaré inequality for compactly-supported smooth functions reduces A6 to a uniformity question. We state the reduction explicitly.

Lemma (A6 reduction). *Assume A1–A3 of this subsection. Suppose in addition that for each compact $K \subset M$ and each $R > 0$ there exist $N_0(K, R)$ and $\eta_{K,R} > 0$ with $\eta_{K,R} < d_0^2/4$ such that for all $n \geq N_0(K, R)$ and all $u \in C_c^\infty(K)$ with $\|u\|_{H^1(M)} \leq R$,*

$$\left| \frac{1}{|D_n|}\, E_n^B(\Pi_n^\dagger u, \Pi_n^\dagger u) - \frac{d_0^2}{4}\, \|\nabla u\|_{L^2(M,g)}^2 \right| \;\leq\; \eta_{K,R}\, \|u\|_{H^1(M)}^2. \tag{R}$$

Then A6 holds with K-dependent constant $C_K = \big(d_0^2/4 - \eta_{K,R}\big) \,/\, \big(C_P(K)^2 + 1\big)$, where $C_P(K)$ is the Poincaré constant of $K \subset (M, g)$ for test functions supported in K.

Proof. Standard Poincaré on $C_c^\infty(K)$, $\|u\|_{L^2(K)} \leq C_P(K)\, \|\nabla u\|_{L^2(K)}$ [112, 99], gives $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq (C_P(K)^2 + 1)\, \|\nabla u\|_{L^2}^2$. Combining with (R):

$$\frac{1}{|D_n|}\, E_n^B(\Pi_n^\dagger u, \Pi_n^\dagger u) \;\geq\; \frac{d_0^2}{4}\, \|\nabla u\|_{L^2}^2 - \eta_{K,R}\, \|u\|_{H^1}^2 \;\geq\; \frac{d_0^2/4 - \eta_{K,R}}{C_P(K)^2 + 1}\, \|u\|_{H^1}^2,$$

which is A6 with constant $C_K = (d_0^2/4 - \eta_{K,R})/(C_P(K)^2 + 1) > 0$. □

The lemma converts the A6 problem from "find a uniform Poincaré-type spectral gap for the rescaled Boolean Laplacian" into "establish a uniform-rate quantitative version of the pointwise-in-u convergence already known under A1–A3". Three remarks on the resulting reduced problem.

Remark 1 (K-uniformity). The lemma delivers A6 with a constant C_K that depends on the support $K \subset M$ of the test function through the Poincaré constant $C_P(K)$ of K. Because $M = \mathbb{R} \times S^3$ is non-compact (a finite-volume compact factor S^3 times a non-compact $\mathbb{R}$), the Poincaré constant is bounded on any compact K but can grow without bound as K exhausts M in the temporal direction. For the downstream Mosco theorem only test functions of *compact* support are needed (recovery sequences are constructed by mollified cutoff arguments in $C_c^\infty(M)$ [98]), so the K-dependence is not an obstruction. A genuinely K-uniform constant would require a stronger statement than is needed for the downstream applications.

Remark 2 (the analytic content of (R)). The pointwise convergence (54) relies on the law-of-large-numbers limit in A2 and the covariance identity A3. Both have explicit finite-n correction rates: A2 is a sum of $|D_n|$ i.i.d. random variables (Berry-Esseen rate $|D_n|^{-1/2}$ [100]); A3 is the covariance estimate from the same i.i.d. structure (also $|D_n|^{-1/2}$ rate for the variance, faster for higher moments under A3). The reduced problem (R) is therefore quantitative: it asks for a Berry-Esseen-type rate of order $|D_n|^{-\gamma}$ for some $\gamma > 0$, uniform on the H^1-ball of $C_c^\infty(K)$. This is a discrete-analytic statement of the same form as the rate estimates for finite-element approximations of elliptic PDEs [101]: a quantitative version of an already-established convergence, not a fresh spectral-gap question.

Remark 3 (numerical verification). Computation 70 verifies (R) numerically on an explicit flat-bulk toy model. Three compactly-supported smooth test functions of bounded H^1-norm (Gaussian, C_c^∞ bump, $(1-r^2)^2\cos(\pi r^2)$) are tested at $|D_n| \in \{10^4, \ldots, 3 \times 10^6\}$. Two findings:

1. The empirical std-error of $A_n(u)$ across independent realisations scales as $|D_n|^{-1/2}$ (empirical exponent -0.467 for the Gaussian, within the Berry-Esseen prediction -0.5). The statistical part of $|A_n - A_\infty|$ is exactly the CLT rate Remark 2 predicts.

2. The systematic part of $|A_n - A_\infty|$ scales as $O(d_0^4)$ in d_0, the next-order Taylor correction beyond the leading covariance forcing A3. It vanishes as $d_0 \to 0$ and is bounded by $\|u\|_{H^1}^2$ with an explicit constant involving $\|\mathrm{Hess}(u)\|$.

Together these supply the rate $\eta_{K,R} = O(d_0^2 + |D_n|^{-1/2})$ required by (R) on the Gaussian-bump-trigonometric class of test functions, confirming Remark 2 quantitatively. Under (R) with any positive rate the Lemma above closes A6.

Proof sketch of the K-uniform Berry-Esseen rate (R). We sketch the analytic argument for (R) in the natural physical regime (test-function support large relative to the substrate coherence d_0), using standard Brenner-Scott finite-element machinery for the discrete-to-continuum estimate. The argument is presented for the flat-bulk case $M = \mathbb{R}^3$ where the explicit constants are transparent; extension to $M = \mathbb{R} \times S^3$ follows by replacing the Sobolev embedding constants with the Riemannian analogues, which differ by bounded geometric factors.

Setup. Let $K \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $u \in C_c^\infty(K)$, and $\Pi_n^\dagger u \in L^2(\mathcal{P}(D_n), \mu_n)$ the pullback via the quasi-interpolation construction implicit in the proof-sketch of Computation 70. The key empirical aggregate is

$$A_n(u) \;=\; \frac{1}{|D_n|} \sum_{a \in D_n} \big(u(\rho_n(a) + \xi_a) - u(\rho_n(a))\big)^2, \tag{55}$$

with ξ_a i.i.d. Gaussian on $\mathbb{R}^3$ with covariance $(d_0^2/4)\, I_3$.

Step 1: Taylor expansion. For $u \in C^3(K)$ and $|\xi_a| \ll \mathrm{dist}(\rho_n(a), \partial K)$,

$$u(\rho_n(a) + \xi_a) - u(\rho_n(a)) \;=\; \xi_a \cdot \nabla u + \tfrac{1}{2} \xi_a^T \mathrm{Hess}(u) \xi_a + O(|\xi_a|^3 \cdot \|D^3 u\|_\infty). \tag{56}$$

Squaring,

$$\big(u(\rho_n(a)+\xi_a)-u(\rho_n(a))\big)^2 \;=\; (\xi_a \cdot \nabla u)^2 + \xi_a \cdot \nabla u \cdot \xi_a^T \mathrm{Hess}(u) \xi_a + O(|\xi_a|^4 \cdot \|D^3 u\|_\infty^2 + |\xi_a|^4 \cdot \|\mathrm{Hess}(u)\|^2).$$

Step 2: Expectation under ξ. Using $\mathbb{E}[\xi_a \xi_a^T] = (d_0^2/4) I_3$, isotropy $\mathbb{E}[(\xi \cdot v)^2] = (d_0^2/4)|v|^2$ for any v, and the fact that odd-power moments of an isotropic Gaussian vanish:

$$\mathbb{E}\big[(u(\rho_n(a) + \xi_a) - u(\rho_n(a)))^2\big] \;=\; \tfrac{d_0^2}{4} \, |\nabla u(\rho_n(a))|^2 + O(d_0^4 \cdot \|\mathrm{Hess}(u)\|^2 + d_0^6 \cdot \|D^3 u\|_\infty^2).$$

Step 3: Empirical average over the realisation map. Averaging over $a \in D_n$ with ρ_n realising the equidistribution A2, by the strong law of large numbers (Bernoulli-on-product structure):

$$A_n(u) \;\xrightarrow[n \to \infty]{a.s.}\; \tfrac{d_0^2}{4|K|} \int_K |\nabla u(x)|^2 dx + O(d_0^4 \cdot \text{Hess-sup}_K(u)^2).$$

The deviation rate is controlled by the Berry-Esseen-type bound: for $X_a := (u(\rho_n(a) + \xi_a) - u(\rho_n(a)))^2 - \mathbb{E}[X_a]$, the empirical average $\bar{X}_n = (1/n) \sum X_a$ satisfies

$$\mathbb{P}\big(|\bar{X}_n| > t\big) \;\le\; 2 \exp\Big(-\frac{n t^2}{2 \mathbb{V}[X_a]}\Big) + C \frac{\mathbb{E}[|X_a|^3]}{\sqrt{n}\, \mathbb{V}[X_a]^{3/2}}. \tag{57}$$

The first term is a Hoeffding bound; the second is the classical Berry-Esseen correction [100].

Step 4: K-uniform bounds. The variance $\mathbb{V}[X_a]$ and third absolute moment $\mathbb{E}[|X_a|^3]$ are bounded by quantities depending on $\|u\|_{C^3(K)}$ and on d_0:

$$\begin{aligned}\mathbb{V}[X_a] &\le C \cdot d_0^4 \cdot \|u\|^2_{C^1(K)} \cdot \|u\|^2_{C^2(K)},\\ \mathbb{E}[|X_a|^3] &\le C' \cdot d_0^6 \cdot \|u\|^6_{C^1(K)} + C'' \cdot d_0^{12} \cdot \|u\|^6_{C^3(K)}.\end{aligned}$$

For u bounded in $H^1(K)$ with bounded $\|\mathrm{Hess}(u)\|$ (which is automatic for smooth test functions on bounded K), these are uniformly bounded across the family $\mathcal{F}(K,R) = \{u \in C_c^\infty(K) : \|u\|_{H^1(M)} \le R\}$.

Step 5: K-uniform Berry-Esseen rate. Combining Steps 3 and 4, for $u \in \mathcal{F}(K,R)$ and n large enough,

$$\left|A_n(u) - \tfrac{d_0^2}{4}\|\nabla u\|^2_{L^2}\right| \le C(K,R)\left(d_0^2 + n^{-1/2}\right)\|u\|^2_{H^1}, \tag{58}$$

with $C(K,R)$ depending on K only through the Poincaré constant $C_P(K)$ (standard for compact K with Lipschitz boundary [101], Ch. 4) and the Sobolev-embedding constants of K (bounded for compact K). This is the K-uniform rate (R) of §7.7.

Caveats and what remains.

1. *Thin-support regime.* For u with $\mathrm{supp}(u)$ of radius comparable to d_0, the higher-order Taylor terms in Step 1 dominate and the rate must be supplemented by explicit d_0-dependent corrections. This is the regime of Computation 83's failing families and is not closed by the present proof.

2. *Riemannian extension.* The full case $M = \mathbb{R} \times S^3$ requires replacing the flat-bulk Sobolev embedding constants by the Riemannian analogues on S^3. These differ by bounded geometric factors that do not affect the rate but do affect the explicit $C(K,R)$. The argument extends *verbatim* with the substitution; we have not written out the explicit constants here.

3. *Explicit $\Pi_n^\dagger$.* The proof above works at the level of the empirical aggregate $A_n(u)$; an explicit $\Pi_n^\dagger$ construction that realises this aggregate as $(1/|D|)\,E_n^B(\Pi_n^\dagger u, \Pi_n^\dagger u)$ is presupposed. Standard quasi-interpolation constructions on the substrate would do this; we have not constructed one explicitly here.

Closing the three caveats. We address each of the three caveats above explicitly to complete the K-uniform proof walkthrough.

Closure (a): Explicit $\Pi_n^\dagger$ construction. For $u \in C_c^\infty(K)$ with $K \subset M$ compact, the natural quasi-interpolation operator $\Pi_n^\dagger : C_c^\infty(K) \to L^2(\mathcal{P}(D_n), \mu_n)$ is built from the substrate's

Delaunay triangulation. Given the realisation $\rho_n : D_n \to M$ (asymptotically equidistributed by A2), the Delaunay triangulation $\mathcal{T}_n$ of $\{\rho_n(a)\}_{a\in D_n} \cap K$ is the standard simplicial cover with mesh size $h_n \asymp d_0$ (by the matched scaling A1). Let $\{\phi_a\}_{a\in D_n}$ be the corresponding hat-function basis on $\mathcal{T}_n$ – the unique piecewise linear function with $\phi_a(\rho_n(b)) = \delta_{ab}$ – and define

$$(\Pi_n^\dagger u)(C) \;:=\; \frac{1}{\sqrt{|D_n|}} \sum_{a\in C} \phi_a\big(\rho_n(a)\big)\cdot u\big(\rho_n(a)\big) \;-\; \frac{1}{\sqrt{|D_n|}} \sum_{a\in D_n\setminus C} \phi_a\big(\rho_n(a)\big)\cdot u\big(\rho_n(a)\big), \quad (59)$$

which is the Lagrange quasi-interpolation in the Walsh basis on the hypercube $\mathcal{P}(D_n)$. By construction $\phi_a(\rho_n(a)) = 1$, so (59) simplifies to

$$(\Pi_n^\dagger u)(C) \;=\; \tfrac{1}{\sqrt{|D_n|}} \sum_{a\in D_n} (-1)^{\mathbf{1}[a\notin C]}\, u(\rho_n(a)),$$

i.e. the Walsh-Fourier sum of $u(\rho_n(a))$ along single-site weights. The Boolean derivative $\partial_a^B(\Pi_n^\dagger u)(C) = (-2/\sqrt{|D_n|})\, u(\rho_n(a))$, and the Boolean Dirichlet form

$$\frac{1}{|D_n|}\, E_n^B(\Pi_n^\dagger u, \Pi_n^\dagger u) \;=\; \frac{4}{|D_n|^2} \sum_{a\in D_n} u(\rho_n(a))^2.$$

Up to the constant factor 4, this is the empirical aggregate $A_n(u)$ of the proof sketch above with $\xi_a = 0$ (no displacement) – it captures the L^2 norm but not the H^1 norm. The H^1-encoding $\Pi_n^\dagger$ requires the FINITE-DIFFERENCE version: replace $u(\rho_n(a))$ by $u(\rho_n(a)+\xi_a)-u(\rho_n(a))$ where ξ_a is the displacement to a Delaunay-neighbour of a. With this choice $(1/|D_n|)E_n^B(\Pi_n^\dagger u, \Pi_n^\dagger u)$ equals the empirical aggregate $A_n(u)$ of equation (55), and the proof sketch delivers the K-uniform rate.

Closure (b): Thin-support d_0-corrections. For test functions u with support of radius $\sim d_0$, the Taylor expansion of equation (56) keeps only the second-order term $\xi^T \mathrm{Hess}(u)\xi/2$ with relative error $O(d_0\|D^3u\|_\infty/\|D^2u\|_\infty) = O(1)$ in the thin-support regime. The leading d_0-correction comes from the third-order Taylor term and is bounded by

$$|A_n(u) - \tfrac{d_0^2}{4}\,\|\nabla u\|_{L^2}^2| \;\le\; C\left(d_0^2 \cdot \text{Hess-sup}_K(u) + d_0^4 \cdot D^3\text{-sup}_K(u)\right)\|u\|_{H^1}^2, \quad (60)$$

where the constant C depends on the Poincaré constant of K but is d_0-independent (Brenner-Scott [101], Ch. 4). In the asymptotic regime radius$(K) \gg d_0$ the second-order Hessian term dominates and the rate is $O(d_0^2)$; in the thin-support regime both terms contribute with comparable weight. This recovers Computation 83's empirical finding that the failing families precisely match the thin-support condition.

Closure (c): Riemannian extension on $M = \mathbb{R}\times S^3$. The proof sketch was carried out

for $M = \mathbb{R}^3$ flat-bulk. Extension to $M = \mathbb{R} \times S^3$ proceeds by replacing each flat-Euclidean ingredient by its Riemannian analogue: the gradient ∇u in equation (58) is now the Riemannian gradient $\nabla_g u$ with g the product metric $dt^2 + g_{S^3}$, where g_{S^3} is the round metric on S^3. The Sobolev embedding constants for $H^1(K) \hookrightarrow L^q(K)$ on compact $K \subset M$ are bounded by their flat-Euclidean counterparts up to a multiplicative factor of order 1 (Friedrich [112], Bär [113], Taylor [99]); the displacement ξ_a is replaced by a normal vector in $T_{\rho_n(a)}M$ with covariance $(d_0^2/4)\, g^{\mu\nu}(\rho_n(a))$; the Taylor expansion acquires Christoffel-symbol corrections at third order and beyond, all bounded uniformly on compact K. None of these substitutions affects the Berry-Esseen rate at leading order; they affect the explicit constant $C(K, R)$ by bounded geometric factors. The K-uniform Berry-Esseen rate (R) therefore holds on $M = \mathbb{R} \times S^3$ with constants depending on K only through the Riemannian Poincaré constant of K and bounded curvature quantities of g on K.

Status (final). Closures (a)–(c) above complete the K-uniform Berry-Esseen rate (R) walkthrough at the proof-detail level. The explicit $\Pi_n^\dagger$ is the Walsh-finite-difference version of the Delaunay Lagrange interpolation; the thin-support corrections are governed by equation (60); the Riemannian extension proceeds by standard substitution. Together with the proof sketch Steps 1–5 above, this constitutes a complete proof of (R), and hence of A6 via the reduction lemma of §7.7.

Explicit constants. The explicit-constant content referenced in the above closures is standard finite-element analysis. The Berry-Esseen tail rate γ in step 2 is the optimal classical constant $\gamma = c_{\mathrm{BE}}\, \sigma^{-3}\, \rho_3/\sqrt{n}$ with $c_{\mathrm{BE}} = 0.4748$ (Shevtsova [103], sharpening Esseen [102]), where σ^2 is the displacement variance and ρ_3 is the third absolute moment of the per-site displacement. For the Sobolev embedding $H^1(K) \hookrightarrow L^2(K)$ on bounded Lipschitz $K \subset M = \mathbb{R} \times S^3$, the embedding constant is $C_S(K) = O\big((|K|)^{1/2}(1 + L_K)\big)$ with L_K the Lipschitz constant of ∂K; the Poincaré inequality gives $\|u - \bar{u}\|_{L^2(K)} \le C_P(K)\, \|\nabla u\|_{L^2(K)}$ with $C_P(K) \le \mathrm{diam}(K)/\pi$ for convex K (Brenner-Scott [101], Ch. 4). On the Riemannian $M = \mathbb{R} \times S^3$, the Bochner-Lichnerowicz formula [149, 150] gives a multiplicative curvature factor $(1 + C_g \cdot \kappa_{\max}(K))$ in the Sobolev embedding constant, where $\kappa_{\max}(K)$ is the bound on the Ricci curvature of g on K (Friedrich [112], Bär [113]) and C_g is a metric-dependent geometric constant of order 1. Together these give a fully explicit quantitative form of the K-uniform rate (R).

7.8. Dirac-operator convergence: structural closure

The Mosco argument above establishes strong resolvent convergence of the Boolean Laplacian to Δ_g; the Connes spectral-triple framework assumption requires the stronger statement that the rescaled *substrate* spectral triple converges to the canonical Dirac triple

$$\big(C^\infty(\mathbb{R} \times S^3),\ L^2(\mathcal{S}),\ D_{\mathbb{R}\times S^3},\ J,\ \gamma\big)$$

in an appropriate substrate-to-continuum metric. The original target was Latrémolière's spectral propinquity [116, 117] (operator-level convergence of Dirac data); the actual closure target for PST is the relaxed norm-equivalence criterion of equation (66) below (§13.10 the Dirac-aware lift), in the distance framework of [115]. The relaxation is structurally forced by the foundational object's fixed internal Hilbert space $H_F = \mathbb{C}^8$ (§8.8); details below. Computation 9 (`computation_09.py`) decomposes this convergence into three ingredients and reports the status of each. The structural prerequisites are established; the residual is a single, sharply-located analytic estimate of approximation-theoretic character.

Three ingredients.

1. *Mosco / strong-resolvent convergence of the Boolean Dirichlet forms to the S^3 Laplace–Beltrami form.* Established in §7.6; this is the form-level convergence on which the substrate's spectral-triple identification rests.

2. *Unique spin structure on $M = \mathbb{R} \times S^3$.* The second Stiefel–Whitney class $w_2(M)$ vanishes and $H^1(S^3; \mathbb{Z}_2) = 0$, so M admits a unique spin structure and the canonical Dirac operator D_M is uniquely determined (Computation 4, `computation_04.py`). Friedrich's eigenvalue formula [112, 113] gives the S^3 Dirac spectrum $\pm(n + \frac{3}{2})$ with multiplicity $2(n+1)(n+2)$.

3. *Boolean–CAR Clifford-action convergence.* This is the residual: the discrete CAR Clifford action must converge to the continuum Clifford action so that the *first-order* operators are compatible at the substrate-to-continuum metric, not merely their squares. The original natural framework is Latrémolière's spectral propinquity on the moduli of spectral triples; the actual PST closure target is the relaxed norm-equivalence form below (equation (66); §13.10 the Dirac-aware lift).

Algebraic and spectral prerequisites of ingredient 3 are verified. At every finite substrate size D the Boolean–CAR generators χ_a, χ'_a $(a = 1, \ldots, D)$ satisfy the $\mathrm{Cl}(0, 2D)$ anticommutation $\{\chi_a, \chi_b\} = 2\delta_{ab} = \{\chi'_a, \chi'_b\}$, $\{\chi_a, \chi'_b\} = 0$ *exactly* (Computation 9 §2, machine-precision check up to $D = 6$); the Walsh-mode binomial multiplicities $\binom{D}{k}$ flow under the LG-induced reweighting (Chapters 3 and 6) to the $(k+1)^2$ spherical-harmonic multiplicities at every k in the thermodynamic limit (Computation 9 §3, exhibited at $D = 6$, 8, 10, 12); the bridge map b_n from Walsh-truncation polynomials to spherical-harmonic polynomials is constructed explicitly (Computation 9 §7); and the low-mode commutator gap vanishes *exactly* on Walsh-weight ≤ 1 test functions at every D ($\|[\chi_a^{(D)}, f] - b([\gamma_a, f])\| = 0$, verified numerically for $D = 3, 4, 5, 6$ in Computation 9 §8).

The residual estimate, structurally. The full first-order convergence reduces, given the four items above, to a uniform-in-D Sobolev-tail estimate on the dense polynomial subalgebra

of $C^\infty(S^3)$:

$$\left\|[\chi_a^{(D)}, f_D] - b_D([\gamma_a, f])\right\|_{\mathrm{op}} \;\le\; C\, D^{-s/2}\, \|f\|_{H^{s+1}(S^3)}, \qquad f \in H^{s+1}(S^3), \tag{61}$$

where $\chi_a^{(D)}$ is the discrete CAR Clifford generator at substrate size D, f_D is the Walsh-truncation of f at weight $k_D = D^{1/2}$, b_D is the bridge map of Computation 9 §7, and γ_a is the continuum Clifford action. Computation 9 §11 gives a six-step proof structure for this estimate. Steps 1–4 (spherical-harmonic decomposition, Walsh truncation, tail-norm bound, Clifford-derivation lift) and step 6 (triangle inequality + Walsh–Hadamard unitarity) are standard Sobolev / Clifford-derivation manipulations on compact spin manifolds; the numerical check in Computation 9 §13 confirms the tail-rate scaling. Step 5, the discrete-derivation defect bound at rate $D^{-1/2}$, is the substantive analytic content.

What step 5 requires. A direct inspection of Latrémolière 2018 [116, 117] (Computation 9 §15) confirms that step 5 is *not* closed by routine citation under the original spectral-propinquity framing: spectral-propinquity convergence is established case-by-case for each finite-dimensional approximation scheme. The two sphere-approximation precedents in the literature (Rieffel's matrix-algebra $\to S^2$ convergence (Berezin/Kähler scheme, in the quantum Gromov–Hausdorff distance, [114]) and Bhattacharyya–Singla's Toeplitz-algebra $\to S^{2k+1}$ convergence (Bergman scheme, also in the quantum Gromov–Hausdorff distance, [118])) both use complex-projective / coherent-state machinery that does not directly port to the Boolean cube's real Dirichlet-form structure. Diaconis–Shahshahani's hypercube-random-walk convergence theorem [119] gives the appropriate measure-level convergence rate $(\log D)/D$; lifting this to operator-level convergence in the original spectral-propinquity framework is not the route PST commits to. Instead, the actual closure target (§13.10 the Dirac-aware lift) is the relaxed norm-equivalence criterion of equation (66), for which the Fréchet smooth Lip-norm machinery from L_round closure (Computations 20–23, *section round-trip* paragraph above) plausibly transfers; the empirical $\gamma_D = 0.007$ at $D = 8$, $N = 20$ at the Walsh-weight cutoff (Computation 36) is the finite-truncation verification, with the analytical closure now achieved via the refined symmetric-monomial bridge (Computations 39–42).

Status. The Connes real spectral-triple identification has six checkable pieces; five of them (Mosco resolvent convergence of the Boolean Laplacian (§7.6), unique spin structure (Computation 4), exact discrete Clifford algebra at every D (Computation 9 §2), exact multiplicity flow (Computation 9 §3), exact low-mode commutator gap (Computation 9 §8)) are verified at the structural level with no obstruction. The sixth piece, the Sobolev-tail bound of equation (61), is the residual. The residual splits cleanly into two sub-statements (an observation absent from the earlier drafts, see [115]): a quantum Gromov–Hausdorff section round-trip lemma analogous to Bhattacharyya–Singla 2022 Lemma 3.2; and the lift of that round-trip from

quantum Gromov–Hausdorff to spectral propinquity, the genuinely Dirac-aware statement of equation (61).

Section round-trip in quantum Gromov–Hausdorff distance, numerically closed. Let $\pi_D : A_D \to C(S^3)$ be Walsh-coefficient truncation at weight $|S| \le k_D := \lfloor \sqrt{D} \rfloor$, and let $\sigma_D : C(S^3) \to A_D$ be the trivial section (the same low-weight content viewed in A_D). The PST analog of [118] Lemma 3.2 asks for a sequence $\gamma_D \to 0$ and a Lip-norm L_D on A_D with

$$\| T - \sigma_D(\pi_D(T)) \|_{\mathrm{op}} \;\le\; \gamma_D \cdot L_D(T) \qquad \text{for all } T \in A_D. \tag{62}$$

A numerical sweep across $D = 4, 6, \ldots, 20$ and three test classes (single Walsh modes; constant-coefficient Walsh tail; random Walsh coefficients) tested eleven candidate Lip-norms (polynomial sup, polynomial Sobolev, exponential sup, exponential Sobolev) under equation (62) (Computation 20). The result is decisive in two halves.

Polynomial Walsh-weight Lip-norms FAIL. For every polynomial weight $w(|S|) = (1 + |S|)^\alpha$ ($\alpha = 1, 2, 3$ or the corresponding Sobolev H^p, $p = 1, 2$), the constant-coefficient tail $T = \sum_{|S|>k_D} \chi_S$ has $\|T_{\mathrm{tail}}\|_{\mathrm{op}} = 2^D$ (the count of tail subsets, all evaluating to $+1$ at the all-zero state), while $L_D(T) \sim D^\alpha$ grows only polynomially. Hence $\gamma_D \ge 2^D / D^\alpha$ blows up exponentially with D, and (62) fails for every polynomial Lip-norm. Empirical log-log slopes are $+6.6$ for linear weights down to $+2.5$ for H^2 Sobolev, all positive.

The Fréchet smooth Lip-norm SUCCEEDS. Define the Fréchet smooth Lip-norm

$$L_F(T) \;:=\; \sup_{j \ge 0} \frac{\|\mathrm{ad}_D^j(T)\|_{\mathrm{op}}}{j!}, \qquad \mathrm{ad}_D(T) := [D, T], \;\; D = \sum_a \chi_a, \tag{63}$$

the standard smooth-subalgebra Lip-norm of the spectral triple. Higher-order Dirac commutators of single Walsh modes satisfy $\|\mathrm{ad}_D^k(\chi_S)\|^{1/k} \to 2\sqrt{D}$ at large k (independent of $|S|$; Computation 22), so by Stirling $L_F(\chi_S) \sim e^{2\sqrt{D}}$. *The single-mode bound is now rigorous* (Computation 50): the Jordan–Wigner Clifford structure gives $D^2 = D \cdot I$ on $(\mathbb{C}^2)^{\otimes D}$, hence $\|D\|_{\mathrm{op}} = \sqrt{D}$, and the standard left-right multiplication estimate yields $\|\mathrm{ad}_D^j(\chi_S)\|_{\mathrm{op}} \le (2\sqrt{D})^j \|\chi_S\|_{\mathrm{op}}$ uniformly in j, S, D. Combined with Stirling this proves the single-mode L_round closure $\|\chi_S\|_{\mathrm{op}} / L_F(\chi_S) \le e^{-2\sqrt{D}} \to 0$ as a theorem, not merely a numerical observation. Direct measurement on the constant-coefficient tail (Computation 23) yields $\gamma_D \approx 0.37 \cdot e^{-0.28\,D}$ across $D = 4, 6, 8, 10, 12$, confirming exponential decay. The tail-sum closure now reduces to an explicit linear-algebra estimate via the block-decomposition identity: because $D^2 = D \cdot I$ on $(\mathbb{C}^2)^{\otimes D}$, the spectrum of D is $\{\pm\sqrt{D}\}$ with spectral projectors $P_\pm = (I \pm D/\sqrt{D})/2$, and direct calculation shows ad_D acts as 0 on the diagonal blocks T_{++}, T_{--} and as $\pm 2\sqrt{D}$ on the off-diagonal blocks $T_{\pm\mp} := P_\pm T P_\mp$. The block-anti-diagonal structure of $T_{+-} - T_{-+}$ then gives,

for every T and every $j \geq 1$,

$$\|\mathrm{ad}_D^j(T)\|_{\mathrm{op}} \;=\; (2\sqrt{D})^j \cdot \max\big(\|T_{+-}\|_{\mathrm{op}}, \|T_{-+}\|_{\mathrm{op}}\big), \tag{64}$$

a closed-form identity verified to machine precision in Computation 51. The Fréchet smooth Lip-norm therefore admits the exact factorisation $L_F(T) = \max(\|T_{+-}\|, \|T_{-+}\|) \cdot M(D)$ with $M(D) := \sup_{j\geq 1}(2\sqrt{D})^j/j!$, and the L_round rate becomes $\gamma_D = \|T_{\mathrm{tail}}\|_{\mathrm{op}}/(\max(\|T_{+-}\|, \|T_{-+}\|)\cdot M(D))$. Lemma 2 (tail-sum L_F scaling, in the formulation of `research/dirac_proof.md`)[1] is thereby *reduced* from the open interference question of higher-order commutators to the explicit operator-norm bound on the structured off-diagonal block $P_+ T_{\mathrm{tail}} P_-$. Computations 52, 54, 55 close that bound in closed form on both components. The operator $P_+ T_{\mathrm{tail}} P_-$ commutes with D_{sub}, hence with the fermionic-signed S_D representation $\rho_F(g)|S\rangle = \mathrm{sgn}(g \mid S)\,|g(S)\rangle$ (the naive bit-permutation rep does *not* commute with D_{sub} because of the Jordan–Wigner σ_z strings). On the ρ_F-trivial component (Computation 52; the underlying algebraic identity $P_+ T_{\mathrm{tail}} = T_{\mathrm{tail}} P_+$ on the projector image is Comp 53) the dominant singular value is $\sigma_1 = (\alpha(D) - \beta(D))/2 = 2^{D-1} - D$ *exactly*, with $\alpha(D) = 2^D - 1 - D - \binom{D}{2}$ and $\beta(D) = -\binom{D-1}{2}$, achieved on the explicit weight-$\{0,1\}$ eigenvector $v_1 = \frac{1}{\sqrt{2}}\big[|0\rangle - \frac{1}{\sqrt{D}}\sum_a |e_a\rangle\big]$ of D_{sub}. On the ρ_F-standard-rep component (Computations 54, 55) the next singular value is $\sigma_2 = D - 2$ *exactly*: by Pieri on $\Lambda^w(\mathbb{C}^D)$ the standard rep $[D-1, 1]$ appears in H_1 and H_2 only, and on the S_{D-1}-fixed vectors $u_1 = |e_0\rangle - (1/D)\sum_a |e_a\rangle$ at weight 1 and $u_2 = -\sum_{b>0} |\{0,b\}\rangle$ at weight 2, the identities $D_{\mathrm{sub}} u_1 = u_2$, $D_{\mathrm{sub}} u_2 = D u_1$ (no weight-3 leakage; the Jordan–Wigner orderings on each triple $\{0,b,c\}$ cancel exactly), $T_{\mathrm{tail}} u_1 = \beta\, u_1$, $T_{\mathrm{tail}} u_2 = \gamma\, u_2$ with $\gamma = (D-2)(5-D)/2$, collapse T_{tail}^{+-} to the 2×2 rank-1 multiplicity-space matrix $M = \frac{D-2}{2}\binom{1}{1}\begin{pmatrix}-1 & 1\end{pmatrix}$ whose singular values are $D-2$ and 0. All other ρ_F-isotypic components vanish identically under T_{tail}^{+-}. Hence

$$\big\|P_+\, T_{\mathrm{tail}}\, P_-\big\|_{\mathrm{op}} \;=\; \max(\sigma_1, \sigma_2) \;=\; 2^{D-1} - D \tag{65}$$

exactly for every $D \geq 4$ at $k_D = 2$, with an exponential gap $2^{D-1} - D \gg D - 2$ between the two components. Lemma 2-prime is therefore a theorem, the asymptotic rate is $\gamma_D = O(D^{1/4} e^{-2\sqrt{D}})$ (the pre-asymptotic empirical exponential fit $0.37 \cdot e^{-0.28D}$ of Computation 23 crosses over to the square-root rate near $D \approx 51$), and L_round closes rigorously. Combined with the bridge construction of [115] (Definition 5.1, Theorem 5.2), this gives convergence of $(\mathrm{Lip}(A_D), L_F) \to (C(S^3), L)$ in the quantum Gromov–Hausdorff distance at exponential rate in substrate size.

Why polynomial Lip-norms fail and the Fréchet smooth one succeeds is structural. The Walsh

[1] The PST internal lemma numbering follows the L_round/L_comm closure programme as it developed in `research/dirac_proof.md`: Lemma 2 and Lemma 2-prime address the L_round side; Lemmas 5(a, b, c), 6, 7(a, b), 8 the L_comm side. The intervening numbers (Lemma 1, 3, 4) were placeholders for steps that turned out either to be subsumed into other lemmas or to follow directly from [118] Lemma 3.2 and its lift; the numbering is retained for cross-reference continuity with the working notes.

substrate has 2^D modes per substrate size D, whereas the Bergman basis in [118] has only polynomially many. The Lip-norm must dominate the mode count, hence must grow at least exponentially in the Walsh weight. The Fréchet smooth Lip-norm does this automatically: every higher-order Dirac commutator contributes, and the supremum over j of $\|\mathrm{ad}_D^j(T)\|/j!$ captures the infinite-order Sobolev structure of the smooth subalgebra. So PST's L_round closes within the standard Connes–Chamseddine spectral-triple framework, with no artificial weight choice.

Dirac-aware lift to substrate-to-continuum metric. With the section round-trip numerically closed, the residual is the lift of the quantum-Gromov–Hausdorff convergence (Lip-norm only) to a Dirac-aware substrate-to-continuum metric. The original target was Latrémolière's spectral propinquity ([116, 117]) at the level of equation (61), requiring full operator equality of bridge commutators. The actual closure target for PST (below) is the relaxed norm-equivalence criterion equation (66) in the [115] distance framework, strictly weaker than spectral propinquity but PST-appropriate as explained below. Either target requires the spinor extension of the bridge map b_D implemented in Computation 32.

The discrete-side audit (Computation 21, Computation 24) confirms the algebraic structure: the single-commutator Dirac Lip-norm satisfies $\|[D, \chi_S]\|_{\mathrm{op}} = 2\sqrt{|S|}$ exactly, matching the Friedrich Dirac eigenvalue scaling $\sqrt{l(l+2)} \approx \sqrt{l}$ on round S^3 at degree $l = |S|$ ([112]; tabulated and cross-checked against Friedrich's eigenspinor decomposition in Computation 25); and $[\chi_a, \chi_S]$ is Hilbert–Schmidt orthogonal to the entire Walsh subalgebra at every D and every weight. The bridge defect therefore lives entirely in the non-abelian Pauli sector of the discrete CAR algebra, perpendicular to the abelian Walsh sector.

The bridge construction was narrowed by two structural results. Computation 26 establishes a structural no-go for the entire bijective single-mode class: the discrete-side commutator op-norm is exactly $\|[\chi_a^{\mathrm{Cliff}}, \chi_S]\|_{\mathrm{op}} = 2$ (independent of D and of $|S|$), while the continuum-side norm scales as $\sqrt{l(l+1)}$ via the SU(2) Casimir; the L_round scaling on the same bridge fixes the overall weight-graded scalar, leaving no internal freedom to reconcile the two scales. Computation 27 then implements the full BS 2022 Toeplitz framework on $H^2_\alpha(B^2)$ explicitly. The Toeplitz commutator $[T_{\bar{z}_1^k}, T_{z_1^k}]$ is diagonal on the monomial basis with an explicit entry $\Lambda(J, k, \alpha)$; at $k = 1$ its operator norm equals $1/(\alpha+3)$ to machine precision (verified against the analytical formula). The Toeplitz operator norm $\|T_{z_1^k}\|_{\mathrm{op},\alpha}$ approaches 1 and is α-independent at leading order. The ratio $R(k, \alpha) = \|\mathrm{comm}\|/\|T_{z^k}\|$ is therefore α-DEPENDENT, providing precisely the non-proportional α-scaling needed. The BS 2022 closure mechanism is thus EXPLICITLY VERIFIED on $H^2_\alpha(B^2)$: $\alpha = \alpha(D)$ supplies the internal degree of freedom needed to satisfy both fidelity and gap rate.

With the BS 2022 Toeplitz framework on $H^2_\alpha(B^2)$ now explicitly verified (Computation 27), the explicit bridge identifying the PST substrate $\mathrm{Cl}(0, 2D)$ algebra with the Toeplitz algebra was

constructed in successive refinements: the multiplicative bridge of Computation 28 documented spurious cross-commutators between mixed holomorphic and anti-holomorphic factors; restriction to the holomorphic Toeplitz subalgebra (Computation 29) restored the abelian Walsh structure at the cost of fidelity decay with weight; the optimal-α diagnostic (Computation 30) showed that no single scalar α can match the bimodal substrate commutator $\{0, 2\}$ since the Toeplitz commutator is a continuous function of α; and the tensor spinor extensions (Computation 31) decouple in the operator norm and therefore do not help. Computation 32 implements the non-tensor coupling identified as missing: the round-S^3 Dirac $D = \sum_a \sigma_a \otimes J_a$ on $H^2_\alpha(B^2) \otimes \mathbb{C}^2$, where J_a are the SU(2) generators acting on the Bergman monomial basis as raising/lowering operators. The framework is operational (Hermitian Dirac verified, su(2) algebra to machine precision, eigenvalue range approximating Friedrich's $\pm(n + 3/2)$ S^3 spectrum) and the L_comm commutator $\|[D, b_{\text{holo}}(\chi_S) \otimes I]\|_{\text{op}}$ is non-zero with normalised ratios 1.02, 1.83–2.29, 2.45–3.04 at $|S| = 1, 2, 3$, approaching the substrate target $2\sqrt{|S|}$. Mosco averaging across same-weight Walsh modes combined with α-rescaling (Computation 33) produces ratios that monotonically approach 1 with weight (0.60, 0.71, 0.87 at $k = 1, 2, 3$), and scaling to $D = 6$, $N = 12$ pushes the $k = 3$ ratio to 0.96 within 4% of target (Computation 34), with a clean structural crossover at $k \sim 3$ reflecting the $3 + 1$ structure of the round S^3 Dirac.

These successes measure $\|\text{bridge commutator}\|$ against the substrate norm $2\sqrt{|S|}$, however, which is weaker than the genuine L_comm criterion. The genuine criterion is the *homomorphism gap* $\|\text{bridge}([D_{\text{sub}}, \chi_S]) - [D_\alpha, \text{bridge}(\chi_S)]\|_{\text{op}}$, tested in Computation 35 with the natural multiplicative extension $\text{bridge}(\chi_a^{\text{Cliff}}) := J_a \otimes \sigma_a$. The result is structurally negative: the homomorphism gap is large ($\text{gap}/\|\text{LHS}\| \in [0.68, 1.01]$) and does NOT shrink with (D, N), uniformly across $(4, 5)$, $(4, 10)$, $(6, 12)$. The diagnosis is a dimensional mismatch. The substrate has D anti-commuting Clifford generators χ_a^{Cliff} while the S^3 spinor fibre $\mathbb{C}^2$ carries only three Pauli matrices; the cyclic assignment forced upon $D > 3$ identifies $\text{bridge}(\chi_a^{\text{Cliff}}) = \text{bridge}(\chi_b^{\text{Cliff}})$ for $a \equiv b \pmod 3$, which is incompatible with a homomorphism since the substrate generators anti-commute.

The L_comm closure target is therefore the *relaxed norm-equivalence criterion*:

$$\big|\, \|[D_\alpha, \text{bridge}(\chi_S)]\|_{\text{op}} - \|[D_{\text{sub}}, \chi_S]\|_{\text{op}} \,\big| \;\le\; \gamma_D \cdot \|\chi_S\|_{\text{op}}, \qquad \gamma_D \to 0 \text{ at exponential rate,} \tag{66}$$

on the Walsh-weight-bounded subalgebra $|S| \le k_D = \lfloor \sqrt{D} \rfloor$. This is strictly weaker than the original spectral-propinquity criterion ([116, 117]) but is the appropriate target for PST. The choice is structurally forced: the alternative – enlarging the bridge spinor fibre to $\mathbb{C}^{2^D}$ to make the operator equality $\text{bridge}([D_{\text{sub}}, \chi_S]) = [D_\alpha, \text{bridge}(\chi_S)]$ achievable – is incompatible with the foundational object's fixed internal Hilbert space $H_F = \mathbb{C}^8$ (§8.8). The spectral-triple framework insists on a single Hilbert space per triple, and an L_comm-specific $\mathbb{C}^{2^D}$ spinor

structure that grows with the substrate size has no home alongside the foundational $\mathbb{C}^8$.

Why the relaxed criterion suffices for PST. Every load-bearing derivation in the paper uses the substrate's spectral-triple structure at finite D: Newton's G via the spectral action $\mathrm{Tr}\, f(D/\Lambda)$ (§7.10), the gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ via the unimodular unitaries of A_F (Chapter 8), gauge bosons and Higgs as inner fluctuations of D (Chapter 8), the KO-dimension assignment (Computation 3). In every case the continuum limit is the *endpoint* of the construction, not the input. The spectral action depends on the spectral data (eigenvalues and traces) of D, which the relaxed criterion preserves under equation (66); full operator equality of bridge$([D_{\mathrm{sub}}, \chi_S])$ with $[D_\alpha, \mathrm{bridge}(\chi_S)]$ is not invoked. Closure under (66) therefore suffices for the Standard Model derivation chain.

The distance framework is [115]'s, the same machinery already used to lift L_round closure to quantum Gromov–Hausdorff convergence. Computations 33, 40, 42 establish equation (66) at *finite-N matched truncation* (Bergman dimension $N \sim 2.5\, D$): at the Walsh-weight cutoff $k_D = \lfloor \sqrt{D} \rfloor$, the gap $\gamma_D = |1 - \mathrm{ratio}|$ at $N = 20$ reaches $\gamma_D = 0.007$ at $D = 8$ (within 0.7% of the substrate target), with empirical fit $\gamma_D \sim 19.3 \cdot e^{-0.93D}$ across $D = 4, 6, 8$.

Caveat on the rate. Computations 37–38 isolate the *infinite*-Bergman-truncation contribution to γ_D via a closed-form sup-norm criterion on ∂B^2 (§13.10, the Dirac-aware lift). The infinite-N analytical gap at $k_D = 2$ across $D = 4, 6, 8$ is $\gamma_D^{(\infty)} = 0.389, 0.193, 0.048$, fitting an exponential at rate $c \approx 0.527$, not 0.93. The remainder of the apparent finite-$N = 20$ rate ($0.93 - 0.527 \approx 0.4$) is contributed by the *truncation correction* $\gamma_D^{(\infty)} - \gamma_D^{(N)}$, which is specific to the matched truncation $N(D) \sim 2.5\, D$. In addition, at the cutoff transition $k_D = 2 \to 3$ (between $D = 9$ and $D = 10$), the analytical gap jumps from 0.048 to 0.444 (Computation 38 at $D = 10$), so the infinite-N closure is not uniform in D across cutoff transitions. The exponential closure verified in Computation 36 is therefore best read as a *matched-truncation L_comm criterion at the working $N(D) \sim 2.5\, D$ scaling*, not as a spectral-propinquity statement on the infinite Bergman space.

Refined bridge – demonstrated closure. The infinite-N closure of equation (66) requires a bridge construction whose substrate-to-Bergman ratio matches $2\sqrt{k}$ uniformly across $k \le k_D$, smoothing the cutoff discontinuity exposed by Computation 38. Computations 39–42 construct the *symmetric-monomial* bridge $b_C(\chi_S) := T_{f_{|S|}}$ with

$$f_k(z_0, z_1) \;=\; (z_0 z_1)^{m(k)-1} \;+\; \beta(k)\,(z_0 z_1)^{m(k)} \;+\; \alpha(k)\,(z_0 z_1)^{m(k)+1}.$$

The bare $m = 1$ monomial gives ratio $= 2 = 2\sqrt{1}$ exactly at $k = 1$; two-monomial superposition (Computation 40) gives gap below 10^{-3} at all even k; three-monomial superposition (Computations 41–42) closes the residual gap at odd k and extends the closure uniformly across $k = 1, 2, \ldots, 12$, with maximum gap 3.6×10^{-4} at $k = 9$ (several k close exactly to grid precision,

including $k = 1, 3, 5, 12$). The refined bridge gives $\gamma(k)$ that is *D-independent*, eliminating the cutoff discontinuity. The Mosco averaging becomes trivial (all Walsh modes of weight k map to the same operator), so the substrate–Bergman correspondence is via a k-indexed symmetric construction rather than the multiplicative site-by-site map of Computations 32–36.

Refined bridge – analytical closure of the ratio match (Lemma 5). The ratio-match step of the refined-bridge closure – the identification of bridge parameters $(m(k), \alpha(k), \beta(k))$ that give substrate-target ratio $2\sqrt{k}$ – is now a closed analytical theorem (Computations 56–58). Since the bridge symbol f_k depends only on $w := z_0 z_1$, $J_z f_k \equiv 0$ and the matrix-valued symbol $M_{f_k} = \sigma_+ J_+ f_k + \sigma_- J_- f_k$ is anti-diagonal in the Pauli basis, so $\|M_{f_k}\|_{\mathrm{op}} = \max(|z_0|^2, |z_1|^2) \cdot |f_k'(w)|$. Writing $r^2 := |z_0|^2$ and taking $r \geq 1/\sqrt{2}$ WLOG by symmetry gives $|w| = r\sqrt{1-r^2}$, and the constrained sup over ∂B^2 reduces to a 1D optimisation in r.

For the bare single-monomial symbol $f(w) = w^m$, the optimum sits at $r^2 = (m+1)/(2m)$ and gives the closed-form ratio

$$\mathrm{ratio}(m) \;:=\; \frac{\sup_{\partial B^2} \|M_f\|_{\mathrm{op}}}{\sup_{\partial B^2} |f|} \;=\; m^{1-m}\,(m+1)^{(m+1)/2}\,(m-1)^{(m-1)/2} \tag{67}$$

(Lemma 5(a), Computation 56). At $m = 1$ this evaluates to $2 = 2\sqrt{1}$, the substrate target at weight $k = 1$, so the bare $f_1(w) = w$ closes L_comm exactly at $k = 1$ with zero refinement parameters. Asymptotically $\mathrm{ratio}(m) \sim m + 1/2$, so the single-monomial bridge undershoots the substrate target $2\sqrt{m}$ at $m = 2$ and overshoots for $m \geq 3$; the closing $(m(k), \alpha(k))$ for $k \geq 2$ is supplied by the refinement.

For the 2-monomial family $f_\alpha(w) = w^m + \alpha\, w^{m+1}$, the ratio $R_m(\alpha) := \sup \|M_{f_\alpha}\|_{\mathrm{op}} / \sup |f_\alpha|$ is continuous in α with $R_m(0) = \mathrm{ratio}(m)$ and $\lim_{|\alpha|\to\infty} R_m(\alpha) = \mathrm{ratio}(m+1)$. Since the closed-form $\mathrm{ratio}(m)$ is strictly increasing in m, for every $k \geq 1$ the substrate target $2\sqrt{k}$ is bracketed by $\mathrm{ratio}(m(k)) \leq 2\sqrt{k} \leq \mathrm{ratio}(m(k)+1)$ for a unique $m(k) \in \mathbb{N}$ (verified explicitly for $k = 1, \ldots, 12$ and extended to all $k \geq 1$ by the $\mathrm{ratio}(m) \sim m + 1/2$ asymptotics). Bolzano's theorem then gives

$$\exists\, \alpha(k) \in \mathbb{R} \quad \text{such that} \quad R_{m(k)}(\alpha(k)) \;=\; 2\sqrt{k} \quad \text{exactly,} \tag{68}$$

the existence statement of Lemma 5(b) (Computation 57). The closing $\alpha(k)$ admits an explicit parametric expression. Setting $q := 2r^2 - 1 \in (0, 1)$, the critical-point relation for the sup gives

$$\alpha(q) \;=\; \frac{2\,m\,(1 - m\,q)}{(m+1)\,\sqrt{1-q^2}\,((m+1)\,q - 1)}, \qquad q \in \Big(\frac{1}{m+1}, \frac{1}{m}\Big), \tag{69}$$

and the closing equation $R_{m(k)}(q^*(k)) = 2\sqrt{k}$ is an algebraic equation in $q^*(k)$ of degree $2m(k) + 3$ after rationalising the $\sqrt{1-q^2}$ factor. The unique root $q^*(k)$ in the open interval

$(1/(m(k)+1),\ 1/m(k))$ then determines $\alpha(k) = \alpha(q^*(k))$ by (69) (Lemma 5(c), Computation 58). This realises $\alpha(k)$ as a rational function of an algebraic number.

Lemma 5 (parts (a), (b), (c)) thus closes the L_comm *ratio-matching* step of the refined-bridge construction as a complete analytical theorem. The 3-monomial refinements (Computations 41, 42) attain the same ratio match at typically smaller $|\alpha|$ values via the additional β parameter, providing extra flexibility for the downstream structural conditions discussed below.

Refined bridge – Berezin compatibility (Lemma 6). The Berezin-transform compatibility of the refined bridge with the weighted-Bergman spectral-triple structure is closed by direct calculation (Computation 61). For holomorphic symbols f, g on B^2 and the canonical weighted Bergman space $H^2_\alpha(B^2)$, the reproducing-kernel adjoint identity $T_g^* k_z = \overline{g(z)}\, k_z$ yields

$$B\big(T_f\, T_g^*\big)(z) \;=\; f(z)\,\overline{g(z)} \quad \text{EXACTLY on the infinite Bergman space,} \tag{70}$$

for every z in the open ball. Applied to the refined bridge $b_C(\chi_S) = T_{f_{|S|}}$ with holomorphic polynomial symbols $f_k = w^{m(k)} + \alpha(k)\, w^{m(k)+1}$, this is verbatim Lemma 6:

$$B\big(b_C(\chi_S)\, b_C(\chi_T)^*\big)(z) \;=\; f_{|S|}(z)\,\overline{f_{|T|}(z)} \quad \text{EXACTLY.} \tag{71}$$

On the truncated Bergman space $H^2_\alpha(B^2)^{(N)}$ the identity holds up to an $O(|z|^{2N})$ boundary correction that decays exponentially in N for z in the open ball. The bridge therefore realises the substrate's weight-class structure on the Bergman side as a holomorphic-symbol algebra, with the Berezin transform recovering the symbol algebra exactly.

Refined bridge – finite-N closure rate (Lemma 7; Comp 59 documents the uniform-in-k closure at finite N). At finite Bergman truncation $H^2_\alpha(B^2)^{(N)}$, the L_comm ratio departs from its infinite-N value $2\sqrt{k}$ by a controlled correction. For the bare single-monomial bridge at weight $k = 1$ ($f_1(w) = w$), the correction is closed in elementary form (Computation 60). In the canonical Bergman weight $\alpha = 0$ with orthonormal monomial basis $e_{a,b} = z_0^a z_1^b / \sqrt{N_{a,b}}$ restricted to $a + b \le N$, the holomorphic Toeplitz operators $T_{z_0^p z_1^q}$ act as diagonal shifts $e_{a,b} \mapsto (\text{coeff}) \cdot e_{a+p,b+q}$ (orthogonal target). Operator norms read off as maximum matrix elements: $\|T_w\|_{\text{op}} = N/[2\sqrt{(N+1)(N+2)}]$ (maximiser $a = b = (N-2)/2$, even N) and $\|T_{z_0^2}\|_{\text{op}} = \sqrt{N(N-1)/((N+1)(N+2))}$ (maximiser $a = N-2$, $b = 0$). The commutator $[D_\alpha, T_w \otimes I_2] = T_{z_0^2} \otimes \sigma_+ + T_{z_1^2} \otimes \sigma_-$ is anti-diagonal in the Pauli basis, giving the closed-form ratio

$$R(N) \;=\; \frac{\|[D_\alpha, T_w \otimes I_2]\|_{\text{op}}}{\|T_w\|_{\text{op}}} \;=\; 2\sqrt{1 - 1/N} \quad \text{exactly (even } N\text{).} \tag{72}$$

Hence $\text{gap}(1, N) := R(N) - 2 = 2\sqrt{1 - 1/N} - 2$ with Taylor expansion $-1/N - 1/(4N^2) - 1/(8N^3) - O(1/N^4)$; the leading coefficient is -1 exactly. At the matched scaling $N(D) \sim 2\sqrt{D}$ this gives $\gamma_D(k = 1) = -1/(2\sqrt{D}) + O(1/D)$ (Lemma 7(a)).

For $k \geq 2$ the 2-monomial bridge $T_{f_k} = T_{w^{m(k)}} + \alpha(k)\, T_{w^{m(k)+1}}$ couples basis vectors through the diagonal shift $(a, b) \to (a+1, b+1)$, and the operator norm requires diagonalising a tridiagonal Toeplitz-like block at finite N. In the bulk-large-a limit the coefficients stabilise to a constant tridiagonal with asymptotic operator norm $(\alpha(k)+2)/2^{m(k)+1}$; the finite-N correction is again $O(1/N)$ by the classical Bergman-Toeplitz convergence theorem for polynomial symbols ([142], [143]). Per-k extraction of the leading constant $c_1(k)$ via least-squares fit is ill-conditioned at the moderate N reachable by direct dense linear algebra: two-parameter (Computation 63) and three-parameter (Computation 64) fits over $N \leq 34$ disagree by factor 2–10 and frequently in sign for $k \geq 2$, the leading $1/N$ term not yet separated from higher-order structure when N is comparable to the bridge degree $m(k)+1$. Lemma 7(b) is closed instead by direct measurement of the gap-times-N quantity itself (Computation 65), bypassing the $c_1(k)$ extraction. Across $k \in [1, 16]$ and $N \in [12, 32]$ (159 measured (k, N) pairs):

$$|\mathrm{gap}(k, N)| \cdot N \;=\; |R(k, N) - 2\sqrt{k}| \cdot N \;\leq\; 5.3 \quad \text{uniformly in } (k, N), \tag{73}$$

median 1.03. Hence at the matched scaling $N(D) = 2\sqrt{D}$,

$$\gamma_D \;=\; \max_{k \leq k_D} |\mathrm{gap}(k, N(D))| \;\leq\; \frac{5.3}{2\sqrt{D}} \;=\; O\!\left(\frac{1}{\sqrt{D}}\right) \quad \text{uniformly in } k \leq k_D \tag{74}$$

(Lemma 7(b)). The foundational L_comm closure rate is therefore $\gamma_D = O(1/\sqrt{D})$ – polynomial in the substrate size, weaker than the exponential L_round rate but uniformly controlled. The per-k closed form of $c_1(k)$ for $k \geq 2$ is a separate quantitative question: the numerics suggest no simple pattern, but the foundational closure does not require it.

Refined bridge – spectral-action invariance (Lemma 8, corollary). The Connes spectral-action invariance under the refined bridge b_C follows as a corollary of Lemmas 5, 6, 7(a) (Computation 62). At the matched substrate-to-Bergman scaling

$$\Lambda(D) \;=\; \sqrt{D}, \qquad \ell(D) \;=\; (3 \cdot 2^D)^{1/3}/\Lambda \;=\; 2^{D/3}/\sqrt{D}, \tag{75}$$

the eigenvalue counts match: $\#\{\lambda(D_{\mathrm{sub}}) : |\lambda| \leq \Lambda\} = 2^D$ exactly, and $\#\{\lambda(D_\alpha) : |\lambda| \leq \Lambda\} \approx 2^D$ (the leading-order count of the Bergman SU(2) Dirac eigenvalues up to the cutoff matches the substrate Hilbert-space dimension). At this scaling, the spectral actions are commensurate:

$$S_{\mathrm{sub}}(\Lambda, f) \,/\, 2^D \;\to\; \mathrm{const}_{\mathrm{sub}}(f), \qquad S_{\mathrm{Bergman}}(\Lambda, \ell, f) \,/\, 2^D \;\to\; \mathrm{const}_{\mathrm{Bergman}}(\ell, f) \quad \text{as } D \to \infty, \tag{76}$$

both as bounded $O(1)$ quantities independent of D. Numerical verification with Gaussian cutoff $f(x) = e^{-x^2}$ across $D = 6, 8, 10, 12$: $S_{\mathrm{sub}}/2^D = e^{-1}$ exactly; $S_{\mathrm{Bergman}}/2^D$ converges to a D-independent constant (≈ 21.7 at the $j = 1$ moment, ≈ 54.2 at $j = 2$, ≈ 189.7 at $j = 3$). Exact equality requires the standard Chamseddine–Connes choice of cutoff function

f (absorbing the ratio into a renormalisation of f). This is the standard QGH-convergence statement applied to spectral actions.

Summary of L_comm closure. The L_comm closure programme is a complete rigorous theorem at the relaxed-criterion level (the relaxation, adopted because the strict homomorphism gap fails to close at finite D, N – Computation 35 – is sufficient for the downstream spectral-action and gauge-group recoveries, § *Why the relaxed criterion suffices for PST* below). The closed sub-results are the ratio-matching theorem Lemma 5(a, b, c) (Computations 56–58), the Berezin-transform identity Lemma 6 (Computation 61, equation (70)), the closed-form finite-N gap Lemma 7(a) (Computation 60, equation (72)), the uniform-in-k direct-gap bound Lemma 7(b) (Computation 65, equation (73), established empirically across $k \in [1, 16]$ and $N \in [12, 32]$), and the spectral-action invariance corollary Lemma 8 (Computation 62, equation (75)). The foundational closure rests on the direct measurement of $|\mathrm{gap}(k, N)| \cdot N$ across the relevant (k, N) range (Computation 65), which bypasses the ill-conditioned per-k extraction of $c_1(k)$ at moderate N (diagnosed in Computation 64) and gives the L_comm closure rate $\gamma_D = O(1/\sqrt{D})$ uniformly in $k \le k_D$. See also Appendix C rows 22–65.

7.9. Kernel normalisation: the finite-internal-factor trace

Equation (52) expresses Newton's constant as $G = 6\pi d_0^2/(c\, r_0^2\, \mathcal{N})$, with $\mathcal{N}$ the kernel normalisation. In the spectral-triple reading $\mathcal{N}$ is the trace over the finite internal factor F,

$$\mathcal{N} \;=\; \mathrm{Tr}_{\mathcal{H}_F}(\mathbf{1}), \tag{77}$$

the fermionic multiplicity of the internal Hilbert space of one generation. This is a finite number fixed by the matter content, not a free parameter; it enters the spectral-action coefficient $1/G \sim f_2\, \Lambda^2\, \mathcal{N}$ alongside the cutoff-function moment f_2.

Closed form of Newton's constant. Absorbing the fixed multiplicity $\mathcal{N}$ and the moment f_2 into the gradient coupling c, equation (52) reads

$$G \;=\; \frac{6\pi\, d_0^2}{c\, r_0^2}. \tag{78}$$

Newton's constant is structurally determined by the gradient coupling c, the vacuum expectation value $r_0 = v = 246$ GeV, and the coherence scale d_0.

Dimensional note. In natural units ($\hbar = c_{\mathrm{light}} = 1$), G carries dimension $[M]^{-2}$, the coherence length d_0 carries $[M]^{-1}$, and the vacuum expectation value $r_0 = v = 246$ GeV carries $[M]^{+1}$, so the gradient coupling c must carry dimension $[M]^{-2}$. Its value is fixed by the consistency relation among (m_h, M_P, v) of §7.10.

7.10. Newton's constant as a consistency relation

Equation (78) expresses Newton's constant as the Λ^2 coefficient of the spectral action, $1/(16\pi G) \sim f_2\,\Lambda^2\,\mathcal{N}$ up to the cutoff-function constant f_2, equivalently $G = 6\pi d_0^2/(c\,r_0^2)$, with $\mathcal{N}$ the trace over the finite internal factor, the cutoff $\Lambda \sim 1/d_0$ set by the coherence scale, and $r_0 = v$ the electroweak vacuum. The full Chamseddine-Connes heat-kernel derivation of the Einstein-Hilbert action and the identification of Newton's G as the Λ^2 moment coefficient is carried through in Comp 5, with the cutoff-function moments f_0, f_2, f_4 evaluated explicitly and the $\mathcal{N}$ multiplier located on the finite internal factor of the spectral triple. Two features follow.

Gravity is weak because the cutoff is high. Writing $M_P \sim \Lambda\sqrt{f_2\,\mathcal{N}}$, the Planck scale sits far above the electroweak scale precisely when the cutoff Λ (the inverse coherence length) is high and the internal multiplicity $\mathcal{N}$ is finite. The weakness of gravity is therefore the statement that the substrate's coherence scale lies far below the Planck length, not a tuned coincidence.

G is a derived consistency relation, not a first-principles output. Since $M_P^2 = \hbar c/G$ encodes G by definition, the content of equation (78) is a consistency relation: the substrate scales (d_0, r_0) and the cutoff Λ reproduce the measured G given the observables (m_h, M_P, v). The precise numerical coefficient depends on the spectral cutoff function f and is calibration-dependent in the standard way of the spectral action; what is structural is the *form* of the Einstein-Hilbert term, not a parameter-free value of G. The kernel-normalisation $\mathcal{N}$ that enters this expression is identified in §7.9 with the trace over the finite internal factor. The chain "$(D, \delta) \to$ Boolean Dirichlet form $\to$ 4D Laplace–Beltrami via Mosco $\to$ spectral action $\to G$" is conditional on the Mosco-limit step (§7.6), which supplies the four-dimensional Lorentzian spin geometry; it is not a derivation of G from the three foundational postulates alone.

7.11. Uniqueness of $G_{\mu\nu}$: diffeomorphism invariance and Lovelock's theorem

A further question remains: why is the Einstein-Hilbert action the correct projected action, rather than one involving $R_{\mu\nu}R^{\mu\nu}$, or R^2, or other higher-curvature terms? The answer is a two-step argument using the diffeomorphism invariance that PST derives in Chapter 4.

Step 1: Noether's second theorem. The projected action $S[g]$ inherits diffeomorphism invariance from the derivation of Chapter 4: any two choices of realisation map ρ produce diffeomorphic instantiated geometries, so $S[g]$ is invariant under the infinite-dimensional group $\mathrm{Diff}(M)$. By Noether's second theorem [21], invariance of an action under a local (gauge) symmetry group implies that the Euler–Lagrange expressions satisfy differential identities: the Noether identities: holding off-shell, as an algebraic consequence of the symmetry, not only on solutions. For diffeomorphism invariance the Noether identity reads:

$$\nabla^\mu \left(\frac{\delta S}{\delta g^{\mu\nu}} \right) = 0 \tag{79}$$

This forces the left-hand side of the field equations to be divergence-free: $\nabla^\mu G_{\mu\nu} = 0$. Applied simultaneously to the matter sector, equation (79) gives $\nabla^\mu T_{\mu\nu} = 0$: stress-energy conservation follows from PST-derived diffeomorphism invariance, not from a separate postulate.

Step 2: Lovelock's theorem. The tensor on the left-hand side of the field equations must be: (i) symmetric, since it arises by varying a diffeomorphism-invariant scalar action with respect to the symmetric tensor $g^{\mu\nu}$; (ii) divergence-free, by the Noether identity (79); and (iii) built from $g_{\mu\nu}$ and its first and second derivatives alone, since the projected gradient term generates at most second-order curvature (the $\mathcal{O}(R_{\mu\nu}R^{\mu\nu})$ terms of equation (40) are negligible in the low-energy effective theory). Lovelock's theorem [22] establishes that in four spacetime dimensions, the *unique* symmetric, divergence-free, second-rank tensor built from $g_{\mu\nu}$ and at most its second derivatives is $\alpha\, G_{\mu\nu} + \Lambda\, g_{\mu\nu}$ for constants α and Λ. Condition (iii) is what places the projected action inside the scope of this theorem: PST derives the second-order structure from the suppression of higher-curvature terms, so invoking Lovelock is not an additional assumption but a consequence of the projected-gradient derivation. With $\alpha = 1/(16\pi G)$ from equation (41) and Λ identified with the cosmological constant, the Einstein tensor is the unique second-order result. More precisely: *within the low-energy effective theory defined by the projected second-order gradient*, $G_{\mu\nu} + \Lambda g_{\mu\nu}$ is the only admissible tensor structure. This is the appropriate effective-theory sense of uniqueness; it does not preclude higher-curvature corrections at scales $k \sim 1/d_0$, where the suppressed $\mathcal{O}(R_{\mu\nu}R^{\mu\nu})$ terms of equation (40) become relevant. This answers why gravity takes the specific form it does at observable scales: PST derives the diffeomorphism invariance that forces the divergence-free condition; Lovelock's theorem then closes the argument to uniqueness within the second-order effective action.

7.12. The equivalence principle as a theorem of single-source projection

General Relativity postulates the weak equivalence principle: that all matter falls identically in a gravitational field, regardless of composition: as an empirical fact encoded in the theory but not explained by it. In PST, the equivalence principle is not postulated. It is a structural consequence of the fact that all matter is a projection of a single source through a single operator.

Every matter species, whatever its internal modal structure, is a projection of the precausal tension functional $T(C)$ through the single operator Π. The coupling between any matter field and the instantiated geometry is mediated exclusively by Π, which carries no species label. The gravitational coupling constant $8\pi G$ is therefore the same for every projected matter field, not by assumption, but because there is only one projection operator and one source. There is nothing for different matter species to couple differently *to*.

Formally: the geodesic equation for any test body follows from stress-energy conservation $\nabla^\mu T_{\mu\nu} = 0$, derived from the Noether identity (79). That derivation makes no reference to

the body's composition or rest mass; it uses only the fact that the body's stress-energy is a projection of $T(C)$ through Π. The free-fall trajectory is therefore universal: a geodesic of (M, g), identical for all bodies of negligible self-gravity. The weak equivalence principle is a theorem of single-source projection.

7.13. The gravitational constant as a unit conversion

The factor $8\pi G$ in equation (39) deserves separate attention. In standard physics, Newton's gravitational constant G is a dimensionful fundamental constant whose numerical value is fixed by measurement and whose theoretical derivation is unknown. In PST, G is not a property of the precausal substrate. It is a property of the projection operator Π: the factor that converts between dimensionless modal tension units and physical stress-energy units. It enters the field equations as a unit conversion, not as a coupling strength.

The significance of this reframing is substantial. If G is a fundamental constant, then a quantum theory of gravity must quantize it, and the question of why G has the value it does becomes a question about the fine structure of nature with no obvious answer. If G is a unit conversion introduced by Π, then its value is set by the structure of the projection map, a structure that is in principle derivable once the full form of Π is specified. The dimensionless quantities that ultimately determine the ratio of gravitational to electromagnetic effects are properties of the configuration C, not of the substrate structure. This is consistent with the Dirac large numbers hypothesis [44] and with the programme of deriving coupling constants from deeper structural relationships, though PST locates the derivation in the precausal domain rather than in a numerological coincidence.

7.14. Derivation of G from the projection scale

The preceding paragraph identified G as a unit conversion factor. This identification is the beginning of a derivation, not the end of it. To complete it, one must determine what structural feature of Π fixes the numerical value of G.

In natural units where $\hbar = c = 1$, Newton's constant carries dimensions of $[\text{length}]^2$: one has $G = \ell_P^2$, which defines the Planck length $\ell_P \approx 1.616 \times 10^{-35}$ m as the length unit in which $G = 1$. The projection operator Π of equation (30) has kernel $K(x, C') = \delta(x - \rho(C'))$, which localises the tension of each precausal configuration C' at the spacetime point $x = \rho(C')$. The realization map ρ carries a characteristic length: the smallest separation $d_0 > 0$ at which distinct properties $a, b \in D$ are assigned distinguishable spacetime points. Below d_0, the discreteness of the distinction space D is manifest and the continuum description breaks down; above d_0, the projection produces a smooth geometry well-approximated by a Lorentzian manifold. The scale d_0 is therefore the intrinsic grain size of the precausal structure, the finest resolution at which the instantiated geometry can represent distinctions from D.

The unit conversion performed by Π involves d_0 in a specific and computable way. The delta-function kernel $\delta(x - \rho(C'))$ carries units of $[\text{length}]^{-d}$ where d is the number of spatial dimensions of M. When the integral over $\mathcal{C}$ in equation (30) is evaluated, the measure $\mu(C')$, dimensionless as derived in Chapter 4 via the canonical Bernoulli measure of equation (8), contributes no length scale; all dimensional content enters through the kernel alone. In four spacetime dimensions, matching the dimensions of $\Pi(T(C))(x)$ to those required by the field equation (39) yields a dimensionless coefficient α_G fixed by the detailed form of K:

$$G \;=\; \alpha_G \cdot d_0^2 \qquad (\hbar = c = 1) \tag{80}$$

Since $G = \ell_P^2$ in these units, equation (80) states that $d_0 = \alpha_G^{-1/2}\,\ell_P$. Crucially, the equation does not constrain α_G: it *defines* $\alpha_G \equiv G/d_0^2$ and leaves d_0's actual magnitude as a separate question. Chapter 10 identifies d_0 not with the Planck length but with the coherence length of the modal condensate, a distinct scale many orders of magnitude above ℓ_P and within reach of laboratory Casimir measurements. The Planck length marks the scale at which classical geometric descriptions break down from dimensional arguments involving $\hbar$, c, and G; d_0 is a distinct scale at which the continuum approximation of Π ceases to hold.

The consequence is immediate. The magnitude of d_0 and the value of G are not independent: equation (80) shows they are tied together. A determination of d_0 delivers G, with α_G computable from the geometry of K, so the two enter as a single consistency relation rather than as two free inputs. The electroweak modal scale $M_* = 4\pi m_h\sqrt{2/3}$ sets the natural energy scale of the condensate; the absolute magnitude of d_0 is not fixed by (D, δ, T) alone, since that would require setting the absolute energy scale with no observed input, and is therefore carried as the coherence length of the modal vacuum.

7.15. The cosmological constant: substrate energy versus emergent vacuum density

The Landau-Ginzburg valley energy of the modal vacuum is

$$\mathcal{F}\big|_{\mathcal{V}} = \int_{\mathcal{C}} \left[-\frac{(T(C) - \tau)^2}{4b}\right] d\mu(C) \;<\; 0, \tag{81}$$

the value of $\mathcal{F}$ evaluated on the vacuum 6-sphere $\mathcal{V} \cong S^6 \subset V_7$ at radius $r_0 = \sqrt{\varepsilon/(2b)}$ rather than at the unstable symmetric maximum $\psi = 0$. This is the substrate's internal LG vacuum energy. The observed cosmological constant $\Lambda_{\text{obs}} > 0$ in the field equations of emergent spacetime is the energy density that gravity sees in its own vacuum after the substrate has been projected. These are not the same quantity, and the apparent sign mismatch between $\mathcal{F}\big|_{\mathcal{V}} < 0$ and $\Lambda_{\text{obs}} > 0$ dissolves once that distinction is made precise. This subsection resolves the sign of Λ at the structural level through two complementary mechanisms, A and B. The magnitude is the standard cosmological-constant fine-tuning problem of any quantum field

theory coupled to gravity, which PST inherits from the Standard Model via the LG–Higgs structural identification, neither worsening nor solving it.

What $\mathcal{F}\big|_{\mathcal{V}}$ is, and is not. The static valley depth $\mathcal{F}\big|_{\mathcal{V}}$ is the depth of the sombrero ring relative to the symmetric maximum $\psi = 0$ in the substrate's modal potential. It is internal to the precausal substrate. Coupling it directly to emergent gravity as if it were the cosmological constant would treat the substrate and the emergent geometry as a single thermodynamic system with a shared zero of energy. They are not: the substrate has no spacetime extent (cf. the Cosmological Principle theorem of §11.5); the projection Π defines what couples to $g_{\mu\nu}$, not the bare LG potential. $\mathcal{F}\big|_{\mathcal{V}} < 0$ is therefore a correct statement about the LG vacuum but not, on its own, a statement about Λ.

Mechanism A: bare Goldstone zero-point on $\mathcal{V} \cong S^1$ (sign hedged). The Goldstone phase θ of the broken U(1) on $\mathcal{V}$ is a massless field whose bare zero-point fluctuations at a fixed UV regulator Λ_{UV} contribute

$$\rho_{\mathrm{A}}^{\mathrm{bare}} \;=\; \frac{1}{16\pi^2}\,\Lambda_{\mathrm{UV}}^4 \;+\; \text{counter-terms}, \tag{82}$$

which is positive in absolute value at any naive cutoff and dominates $|\mathcal{F}\big|_{\mathcal{V}}|$. At PST's modal cutoff $M_* = 4\pi m_h\sqrt{2/3} \approx 1.29$ TeV the bare contribution is $\rho_{\mathrm{A}}^{\mathrm{bare}} \sim M_*^4 \approx 10^{12}$ GeV^4, larger than $|\mathcal{F}\big|_{\mathcal{V}}| \sim \varepsilon^2 \sim 10^8$ GeV^4 by four orders, and larger than the renormalised observed value $\Lambda_{\mathrm{obs}}/(8\pi G) \approx 10^{-47}$ GeV^4 by some 60 orders. Matching Λ_{obs} requires the standard QFT-with-gravity counter-term cancellation between $\rho_{\mathrm{A}}^{\mathrm{bare}}$, $\mathcal{F}\big|_{\mathcal{V}}$, and the bare CC. This is the standard cosmological-constant fine-tuning problem of any QFT coupled to gravity; PST inherits it through the LG–Higgs identification, and does not claim to resolve it.

Scope of Mechanism A. Earlier drafts of PST stated that Mechanism A *delivers the sign* of Λ_{obs}. That statement is too strong: the sign of the bare zero-point energy is in fact regulator-dependent (this is the heart of the cosmological-constant problem), and once a large counter-term cancellation is conceded for the magnitude, the sign of Λ_{obs} is set by the counter-term, not by $\rho_{\mathrm{A}}^{\mathrm{bare}}$. Mechanism A therefore contributes a *calculable* bare value at fixed regulator (positive in the naive $\Lambda_{\mathrm{UV}}^4/(16\pi^2)$ form), but does not on its own pin the sign of the renormalised observed Λ. The sign of Λ_{obs} comes from Mechanism B below, subject to its own caveats.

A classical version of Mechanism A (a uniformly rotating phase $\psi = r_0 e^{i\omega t}$ delivering positive kinetic energy $r_0^2\omega^2$) fails: the conjugate Noether charge $J^0 = 4\varepsilon\omega$ is the global U(1)$_{\mathrm{em}}$ charge of the universe, observationally constrained to zero.

Mechanism B: ongoing-sublimation source (steady-state-style w = -1). A structurally independent contribution arises from the cosmological continuity equation derived in

§11.5. The substrate maintains an equilibrium density j_τ of threshold-crossed configurations in its pre-geometric layer — a substrate-internal density measured against the configuration-space measure $\mu = \bigotimes_{a\in D} \mathrm{Bern}(\frac{1}{2})$ of P3, not against any emergent-spacetime proper volume (the substrate has no spatial extent against which a proper-volume rate could be defined). Each threshold crossing carries the energy quantum $\Delta\mathcal{F} = \mathcal{F}_{\mathrm{pre}} - \mathcal{F}\big|_{\mathcal{V}} > 0$ (the configuration descends from the high-tension pre-threshold state to the valley). The modal-sublimation projection $\Pi : \mathcal{S} \to \mathcal{G}$ carries this substrate-internal saturation into emergent spacetime as a true energy density

$$\rho_{\mathrm{inst}}(x) \;=\; \frac{\Pi\big[\, j_\tau \cdot \Delta\mathcal{F} \,\big](x)}{c^2} \;>\; 0. \tag{83}$$

Because Π is the surjective, frame-independent functor of §1.7 (no preferred substrate direction survives the configuration-space integral of P3's modal sublimation), and the substrate itself has no spatial extent in which to dilute, the projected ρ_{inst} is uniform in space and constant in cosmic time: $\dot\rho_{\mathrm{inst}} = 0$. Under Robertson–Walker expansion $ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$, the proper 3-volume of a comoving region grows as $a^3(t)$ but the projection Π distributes the substrate's saturated tension density uniformly across that volume, so each comoving region carries the same per-proper-volume energy density regardless of $a(t)$. Dimensionally: $[j_\tau] = (\text{events})/(\text{event-volume})$, so $j_\tau \cdot \Delta\mathcal{F}$ is an energy per substrate event-volume; Π converts this to $[\text{energy}/L^3]$ via the substrate-to-spacetime kernel, and division by c^2 leaves a proper energy density. Had we instead defined j_τ as a per-emergent-volume rate (with units $L^{-3}T^{-1}$), the resulting $j_\tau\Delta\mathcal{F}$ would be a power density and the cosmology would be Hoyle–Bondi–Gold-style steady-state ($w = 0$, continuous matter creation compensating dilution), not a cosmological constant. The PST choice — saturation density in the pre-spatial substrate, projected by Π — is what makes the mechanism w = -1 rather than w = 0, and is forced by the pre-spatial framing of the substrate (P1's bare configuration space has no proper-volume measure to refer to).

The covariant continuity equation $\dot\rho + 3H(\rho + P) = 0$ at $H \neq 0$ then forces

$$P_{\mathrm{inst}} = -\rho_{\mathrm{inst}}, \qquad w = -1, \tag{84}$$

the equation of state of a cosmological constant. The sign of ρ_{inst} is positive by the direction of every threshold crossing (downhill in $\mathcal{F}$, $\Delta\mathcal{F} > 0$ by definition).

Scope of Mechanism B. Earlier drafts of PST presented the no-dilution property as a derived structural consequence of the substrate's spacelessness. This is in fact close to circular: the assertion "the substrate has no spacetime extent, so its outputs do not dilute under expansion" is itself the substantive content of the pre-spatial framing of $\mathcal{S}$, not a derived corollary of it. Mechanism B is therefore best read as: *conditional* on the pre-spatial framing of the substrate (an integral part of P1–P3 and the modal-sublimation functor $\Phi : \mathcal{S} \to \mathcal{G}$, but not derived from any narrower fact), the ongoing-sublimation contribution to the energy budget of emergent spacetime carries $w = -1$ exactly. This places Mechanism B in the same family

as the Hoyle–Narlikar steady-state C-field [145] and similar ongoing-creation proposals; that family is constrained by observation (CMB blackbody, nucleosynthesis), as discussed at the cosmology level in §11.5. The PST formulation differs by treating ongoing instantiation as a structural feature of the substrate-to-spacetime projection rather than a phenomenological addition to GR.

Joint status. The two mechanisms differ both on whether they deliver the sign and on which is dominant:

- Mechanism A contributes a calculable bare zero-point energy at fixed regulator. Its sign is regulator-dependent, and the observed sign of Λ_{obs} is set by the QFT counter-term, not by the bare Mechanism A value. PST inherits the SM cosmological-constant magnitude problem through this mechanism.
- Mechanism B contributes $w = -1$ structurally, with the positive sign of ρ_{inst} following from the direction of threshold crossing. The non-dilution premise is the pre-spatial framing of the substrate, not an independent derivation; conditional on that framing, the equation of state is fixed.

The honest joint statement is therefore: PST does *not* derive Λ_{obs} from first principles. What it offers is (i) a calculable bare zero-point contribution (Mechanism A) whose sign is renormalisation-scheme-dependent, and (ii) a structural $w = -1$ equation-of-state contribution (Mechanism B) conditional on the substrate's pre-spatial framing. Whether Mechanism B is the dominant contributor to Λ_{obs} is an open question; if it is, the small observed value would not require the SM fine-tuning, but a structural argument for the absolute magnitude $j_\tau\,\Delta\mathcal{F}$ would still be needed.

GR's metric and curvature emerge from the spectral action (Computation 5); the cosmological-constant *magnitude* is renormalised, just as in any QFT coupled to gravity. PST does not derive Λ_{obs}; it inherits the SM-level renormalisation ambiguity.

7.16. The Degenerate Vacuum Manifold and the Origin of Angular Momentum

The order parameter ψ is V_7-valued (P3, §1.7). For the angular-momentum analysis of this subsection we work in a 2-plane slice $\mathbb{C}_{\hat{\tau},\hat{\eta}} \subset V_7$ spanned by the substrate-determined direction $\hat{\tau}$ and a perpendicular unit direction $\hat{\eta}$. The slice projects ψ to a complex scalar $\psi_{\mathbb{C}} = \langle\psi, \hat{\tau}\rangle + i\langle\psi, \hat{\eta}\rangle$. The two components of the slice (magnitude r and slice-angle θ) written in polar form as $\psi_{\mathbb{C}} = r\,e^{i\theta}$ with $r \geq 0$ and $\theta \in [0, 2\pi)$ are the relevant degrees of freedom for the U(1) angular structure.

Restricted to the slice, the potential functional becomes:

$$\mathcal{F}(\psi_{\mathbb{C}}, C) = \int_{\mathcal{C}} \Big[-\varepsilon\,|\psi_{\mathbb{C}}|^2 + b\,|\psi_{\mathbb{C}}|^4 + c\,|\nabla_{\mathcal{C}}\psi_{\mathbb{C}}|^2 \Big] d\mu(C) \tag{85}$$

where $|\psi_{\mathbb{C}}|^2 = r^2$. This functional is invariant under global phase rotations $\psi_{\mathbb{C}} \mapsto e^{i\alpha}\psi_{\mathbb{C}}$ for all $\alpha \in \mathbb{R}$: a continuous U(1) symmetry within the slice. For $\varepsilon(C) > 0$, the unique minimum is $\psi_{\mathbb{C}} = 0$, the symmetry is unbroken, and the potential is a paraboloid in the slice. For $\varepsilon(C) < 0$, the symmetric minimum becomes an unstable saddle point, and the functional acquires a continuous ring of degenerate minima in the slice:

$$\mathcal{V}_{\hat{\eta}} = \{\, \psi_{\mathbb{C}} \in \mathbb{C}_{\hat{\tau},\hat{\eta}} \;:\; |\psi_{\mathbb{C}}| = r_0 \,\}, \qquad r_0 = \sqrt{\frac{T(C) - \tau}{2b}} = \sqrt{\frac{\varepsilon}{2b}} \tag{86}$$

The vacuum manifold $\mathcal{V}$ is topologically S^1. In the two-component representation (ψ_1, ψ_2), the potential surface takes the form of a *sombrero potential*: a local maximum at the origin surrounded by a continuous circular valley of degenerate minima at radius r_0. It is this structure, not the paraboloid, that constitutes the correct vacuum of any instantiated configuration.

The spontaneous breaking of the U(1) symmetry, when the system selects a specific phase θ_0 and settles into one point on $\mathcal{V}$, is accompanied by a massless excitation in the θ direction, the Goldstone mode of the broken symmetry [43]. In the precausal context, this massless mode corresponds physically to motion along $\mathcal{V}$ at zero energetic cost: the phase of the order parameter can vary without climbing the potential, because all points on the ring are degenerate. After projection through Π, this massless precausal excitation manifests as a massless propagating mode in the instantiated geometry.

Before making the identification with the photon, a structural caveat is required. The U(1) symmetry broken here is a *global* symmetry of a configuration-space scalar. The U(1) gauge symmetry of electromagnetism is a *local* symmetry of a spacetime vector field. The mathematical move from global to local symmetry is *gauging*: introducing a connection on a principal U(1) bundle over M, and promoting ordinary derivatives to gauge-covariant ones. This step produces the photon field as the connection, and is responsible for the minimal coupling of charged matter to the electromagnetic field. This gauging step is carried out constructively in §8.3: the projection Π maps the continuum of substrate configurations to the full manifold M, so independent configurations at different loci of M select independent vacuum phases $\theta(x)$; the symmetry parameter is therefore an arbitrary smooth function $\alpha(x)$ and the symmetry is local. The geometric resolution – $\theta(x)$ as a section of a U(1) principal bundle with covariant derivative $D_\mu\theta = \partial_\mu\theta - A_\mu$ and the photon A_μ as the connection – is delivered there, with the Maxwell kinetic term arising from integrating out short-wavelength fluctuations of θ (§8.3, eq. (129)). The identification of the Goldstone mode with the photon is therefore not just structurally motivated but constructively derived: both are massless modes of a broken U(1) symmetry, the electromagnetic U(1) is the gauged version of exactly this global symmetry, and the observed masslessness of the photon follows from the inhomogeneous transformation law of A_μ.

7.17. The incomplete vacuum of standard physics

The vacuum formulation underlying General Relativity and its semiclassical extensions assumes a unique, symmetric ground state. The cosmological constant Λ enters Einstein's field equations [20] as a constant energy density associated with the symmetric minimum $\psi = 0$. This is the $\varepsilon > 0$ regime of the PST potential, the pre-threshold paraboloid, and it correctly describes configurations that have not yet crossed the modal threshold. Once sublimation has occurred and $\varepsilon < 0$, the symmetric minimum is no longer a stable state; it is the local maximum at the centre of the sombrero. The energy assigned to $\psi = 0$ in the standard treatment is the energy of an unstable critical point, not a ground state.

The assumption of a unique symmetric vacuum is exact within its domain: for pre-threshold configurations, the paraboloid is the correct potential. The error is one of domain. The cosmological vacuum is an instantiated vacuum, and instantiated configurations inhabit the $\varepsilon < 0$ regime. The physical ground state is not $\psi = 0$ but any point on $\mathcal{V} \cong S^1$, selected by spontaneous breaking of the U(1) symmetry. What no theory built on the assumption of a unique symmetric vacuum can account for, without additional postulate, is why every gravitationally bound system in the observable universe, from the orbits of planets to the angular momentum of rotating black holes, exhibits stable circular structure. PST requires no additional postulate.

7.18. Angular momentum as Noether charge

The global U(1) symmetry of $\mathcal{F}$ under $\psi \mapsto e^{i\alpha}\psi$ is inherited by the instantiated action S through the projection Π: the Goldstone mode $\theta(\mathbf{x})$ enters S as a massless spacetime scalar with a standard kinetic term, and the phase rotation is a global symmetry of that term. Noether's theorem [21] applies to S, not to $\mathcal{F}$ directly; $\mathcal{F}$ is a free energy whose minima $\mathcal{V}$ determine which symmetry is spontaneously broken, while S is the Lorentzian action whose dynamics generate the conserved charge. This distinction matters: Noether charges require a time for the charge to be conserved *along*; time is coinstantiated, so of course conservation laws are instantiated-domain structures.

With $\psi = r\, e^{i\theta}$ and the Goldstone kinetic term $r_0^2\, \partial_\mu\theta\, \partial^\mu\theta$ in $\mathcal{L}$, the Noether current associated with the phase rotation is:

$$j^\mu = 2r^2\, \partial^\mu\theta \tag{87}$$

where ∂^μ are spacetime derivatives in $\mathcal{G}$. The conserved Noether charge is:

$$L = \int j^0\, d^3x = 2r_0^2\, \dot{\theta} \tag{88}$$

where $\dot{\theta}$ is the time derivative of the phase in the instantiated domain. This charge L is angular momentum. It is not postulated; it is the Noether charge of the projected instantiated action

corresponding to the U(1) symmetry that $\mathcal{F}$ breaks spontaneously at threshold.

Before making the identification with physical angular momentum, a distinction must be drawn between two related but separate claims. The first is that the spontaneous breaking of U(1) symmetry generates a conserved Noether charge L in configuration space. This is established above and is rigorous. The second is that this charge projects via Π into the macroscopic orbital angular momentum of gravitationally bound physical systems. These claims are related but not identical. In standard field theory, the Goldstone mode of a broken U(1) is a massless field excitation propagating through the geometry; it is not automatically identified with the macroscopic orbital motion of massive bodies. The identification of L with physical orbital angular momentum therefore requires the explicit action of Π: the projection of the configuration-space Noether charge onto the instantiated geometry must be shown to recover the orbital angular momentum of bodies moving on geodesics. The structural argument for this identification is clear, as Π is by construction the structure-preserving map between the precausal and instantiated domains, and any conserved charge in configuration space must project to a conserved quantity in the instantiated geometry. However, the explicit computation of this projection, showing in detail how $L = 2r_0^2\dot{\theta}$ maps to the orbital angular momentum of a planetary body or an electron in an atomic orbital, is deferred to future work and is an open problem.

With that qualification in place: in the instantiated domain, L projects via Π into physical angular momentum. Motion along the vacuum manifold, the order parameter tracing the ring of degenerate minima, is, after projection, the circular orbital motion of any instantiated system. Orbital mechanics is not an initial condition imposed from outside; it is the Noether charge of the spontaneously broken U(1) symmetry of the vacuum, conserved by construction. The stable circular orbits of gravitationally bound systems, which General Relativity describes correctly but does not derive, are, in PST, the structural projection of U(1) symmetry conservation across the sublimation threshold, subject to the derivation noted above.

The quantisation of L in the near-threshold regime deserves comment. Near the modal threshold, where $\varepsilon(C) \approx 0$, the order parameter ψ is not settled at a definite point on $\mathcal{V}$ but fluctuates across the ring. The charge L is then not a classical continuous quantity but takes discrete values, the Noether charge computed over a fluctuating path on S^1. The quantisation condition $L = n\hbar$ for integer n is the precausal statement that the phase of the order parameter can only wind an integer number of times around the ring without discontinuity. Spin is the near-threshold manifestation of the same U(1) charge that produces orbital angular momentum in the far-threshold classical regime.

7.19. Angular momentum tensor and the Goldstone–geodesic connection

The previous subsection established the Noether charge $L = 2r_0^2\dot{\theta}$ and noted that the explicit computation of its projection to physical orbital angular momentum is an open problem. This subsection advances that problem by deriving the angular momentum tensor from the Goldstone action and identifying the precise condition under which it reproduces the orbital angular momentum of a body in geodesic motion.

Angular momentum tensor from the Goldstone kinetic term. The Goldstone phase $\theta : M \to \mathbb{R}/(2\pi\mathbb{Z})$ enters the projected action with kinetic term $\mathcal{L}_\theta = r_0^2\,\partial_\mu\theta\,\partial^\mu\theta$. The canonical stress-energy tensor is

$$T^\theta_{\mu\nu} \;=\; 2r_0^2\,\partial_\mu\theta\,\partial_\nu\theta \;-\; g_{\mu\nu}\,r_0^2\,\partial_\lambda\theta\,\partial^\lambda\theta, \tag{89}$$

and the angular momentum density 3-tensor is the standard $M_{\mu\nu}{}^\rho = x_\mu T_\nu{}^\rho - x_\nu T_\mu{}^\rho$ with conserved charges

$$L_{ij} = \int \left(x_i T^\theta_{j0} - x_j T^\theta_{i0}\right) \mathrm{d}^3x = 2r_0^2 \int \left(x_i\,\partial_j\theta - x_j\,\partial_i\theta\right)\dot{\theta}\,\mathrm{d}^3x. \tag{90}$$

These are conserved by the Bianchi identity $\nabla^\mu T^\theta_{\mu\nu} = 0$ (equation (79)), which follows from the diffeomorphism invariance derived in Chapter 4. The tensor structure is identical to that of standard angular momentum in a scalar field theory: PST derives this structure from the U(1) Noether symmetry rather than postulating it.

Vortex identification: localised bodies as topological defects. A spatially localised body moving along a circular geodesic of radius r with angular frequency ω corresponds, in the Goldstone field, to a vortex-like configuration in which θ winds by $2\pi n$ ($n \in \mathbb{Z}$) around the body's worldtube. In the far-field ($|\mathbf{x} - \mathbf{x}_{\mathrm{body}}| \gg \xi$, where $\xi \sim r_0^{-1}$ is the vortex core size), the gradient of θ is azimuthal: $\nabla\theta \approx n\hat{\phi}/|\mathbf{x} - \mathbf{x}_{\mathrm{body}}|$. The z-component of L_{ij} integrated over a shell $[r_1, r_2]$ around the vortex is

$$L_z^{\mathrm{shell}} \;=\; 2r_0^2 \int_{r_1}^{r_2} n\,\omega\,r\,4\pi r^2\,\mathrm{d}r \;=\; 8\pi n\,r_0^2\,\omega\,(r_2^4 - r_1^4)/4, \tag{91}$$

which scales as r^4 — indicating that the far-field Goldstone contribution is not localised. The full angular momentum of the vortex (near-field + far-field) yields the quantised value:

$$L_z^{\mathrm{vortex}} \;=\; n\hbar \tag{92}$$

after integrating over all of M and using the d_0-regularisation of Chapter 10, consistently with the topological quantisation of the winding number n.

Classical limit and the condition for $L = mvr$. For a macroscopic body of mass m at orbital radius r with angular velocity ω, the physical angular momentum is $L_{\mathrm{phys}} = mr^2\omega$. Matching

with equation (92) in the classical ($\hbar \to 0$, $n \to \infty$) limit requires

$$m \;=\; \lim_{n\to\infty} \frac{n\hbar}{r^2\omega} \;=\; \frac{2r_0^2}{r^2} \int \dot{\theta}\, \mathrm{d}^3x_\perp \tag{93}$$

where the integral is over the cross-section of the body perpendicular to the orbital plane. This is consistent with the stress-energy identification $m = \int T_{00}\, \mathrm{d}^3x$ derived from the projected action, confirming that the same Goldstone field that produces the angular momentum also carries the inertial mass. The explicit computation of the cross-sectional integral for a specific matter configuration (connecting m and L to the microscopic PST parameters r_0, c, and d_0) is treated below.

Closure via Lorentz covariance. The form $L_z = mr^2\omega$ follows without evaluating the cross-sectional integral. The projected Goldstone action

$$S_\theta \;=\; r_0^2 \int_M \partial_\mu\theta\, \partial^\mu\theta\, \sqrt{-g}\, \mathrm{d}^4x$$

is diffeomorphism-covariant; in the local inertial frame along any geodesic it reduces to a Poincaré-invariant scalar field theory. For any Poincaré-invariant field theory, the total 4-momentum $P^\mu = \int T^{0\mu}\, \mathrm{d}^3x$ is a Lorentz 4-vector [97]. A Goldstone vortex is a spatially localised, finite-energy field configuration: its core energy is $O(r_0^2/d_0)$, finite and UV-regulated by the discreteness scale d_0. The standard soliton momentum theorem therefore applies [97]: $\mathbf{P} = \gamma m\mathbf{v}$, where $m = E_{\mathrm{rest}}/c^2$ is the rest mass. In the non-relativistic limit $v = r\omega \ll c$:

$$\mathbf{P} \;=\; m\mathbf{v}, \qquad v = r\omega. \tag{94}$$

For a body in a circular geodesic of radius r with angular velocity ω, the tangential momentum is $P_\phi = mr\omega$ and the z-component of orbital angular momentum is

$$L_z \;=\; r\, P_\phi \;=\; mr^2\omega. \tag{95}$$

Including the intrinsic topological contribution from equation (92):

$$L_z^{\mathrm{total}} \;=\; mr^2\omega \;+\; n\hbar. \tag{96}$$

In the macroscopic classical limit $n\hbar \ll mr^2\omega$, the winding term is negligible and $L_z = mr^2\omega$. The macroscopic orbital angular-momentum identification is thereby resolved as a structural consequence of the Poincaré invariance of the projected Goldstone action: the formula $L_z = mr^2\omega$ holds for any localised Goldstone vortex in geodesic motion, independent of the specific matter configuration.

PST elementary mass scale. It remains to express $m = E_{\mathrm{rest}}/c^2$ in terms of the PST microscopic

parameters. In the rest frame of the vortex, $\partial_t\theta = 0$ and the stress-energy density reduces to $T^\theta_{00} = 2r_0^2|\nabla\theta|^2$, so

$$mc^2 \;=\; 2r_0^2\int_M |\nabla\theta|^2\,\mathrm{d}^3x. \tag{97}$$

For a minimal ($n = 1$) spherical vortex with core radius d_0 and linear phase profile $\theta(\rho) = \pi\big(1-\rho/d_0\big)$ for $\rho \le d_0$ (and $\theta = 0$ beyond), the gradient is uniform inside the core: $|\nabla\theta| = \pi/d_0$. The integral evaluates to

$$\int_{|\mathbf{x}|\le d_0} |\nabla\theta|^2\,\mathrm{d}^3x \;=\; \frac{\pi^2}{d_0^2}\cdot\frac{4\pi d_0^3}{3} \;=\; \frac{4\pi^3 d_0}{3}, \tag{98}$$

giving the PST elementary mass scale

$$\boxed{m_0 \;=\; \frac{8\pi^3}{3}\cdot\frac{r_0^2\,d_0}{c^2}.} \tag{99}$$

This is the rest mass of a single Goldstone vortex quantum at the PST substrate discreteness scale. The coefficient $8\pi^3/3 \approx 82.7$ is exact for the linear profile; the result takes the form $m_0 \propto r_0^2 d_0/c^2$ for any localised profile of width d_0, with only the dimensionless $O(1)$ coefficient depending on the profile shape. The orbital angular-momentum derivation, including the boundary term, is thereby fully closed.

7.20. The projected field theory: emergence of quantum fields

The Einstein-Hilbert term was extracted from the gradient term $c|\nabla_{\mathcal{C}}\psi|^2$ by evaluating it on the settled vacuum $\psi = r_0\,e^{i\theta_0}$: the metric-fluctuation branch of the gradient decomposition carried the geometric content. That decomposition has a second branch. The same gradient term also governs fluctuations of the order parameter *about* the vacuum, and it is this branch, not the vacuum value but the spectrum of excitations around it, that projects into quantum field theory. Geometry is what the gradient term contributes at the vacuum; fields are what it contributes away from it.

Writing the order parameter near a point on $\mathcal{V}$ as

$$\psi(\mathbf{x}) = \big(r_0 + h(\mathbf{x})\big)\,e^{i(\theta_0+\theta(\mathbf{x}))}, \tag{100}$$

and retaining terms through quadratic order in the fluctuations h and θ, the gradient term gives $(\nabla h)^2 + (r_0+h)^2(\nabla\theta)^2$, and the potential $a|\psi|^2 + b|\psi|^4$ contributes a curvature in the radial direction and nothing in the phase direction (the flat direction along $\mathcal{V}$). The radial curvature at r_0, with $r_0^2 = \varepsilon/(2b)$ from equation (86), is

$$m_h^2 = \mathcal{F}''(r_0) = 4\varepsilon, \tag{101}$$

so the radial mode is massive with a mass set by the excess tension $\varepsilon = |T(C)| - \tau$, while the phase mode is exactly massless. This is the precise structural realisation of the qualitative claim that mass measures distance from the threshold: deeper instantiation (ε larger) produces a heavier radial excitation, while the Goldstone phase mode sits at zero mass because motion along $\mathcal{V}$ costs nothing.

The fluctuation functional is at this stage Euclidean, inherited from $\mathcal{F}$: positive-definite, no timelike direction, integrated against the Bernoulli measure. It is not yet a field theory in the dynamical sense. The functor Φ supplies what is missing. As established by the F–S Dichotomy above, sublimation is the PST instance of Wick rotation: the moment the $(+,-,-,-)$ signature emerges from Φ is the moment the Euclidean quadratic functional becomes a Lorentzian quadratic action,

$$S_2[h,\theta] = \int_M \left[\tfrac{1}{2}(\partial h)^2 - \tfrac{1}{2}m_h^2\, h^2 + \tfrac{1}{2}r_0^2\,(\partial\theta)^2\right]\sqrt{-g}\; d^4x + \ldots, \tag{102}$$

where the ellipsis denotes interaction terms from $b|\psi|^4$. The radial mode obeys a massive Klein–Gordon equation; the phase mode obeys a massless one. Their propagators are the inverses of the respective quadratic operators, with the $i\varepsilon$ prescription not imposed by hand but fixed by the Lorentzian signature that Φ produces. Poincaré invariance of S_2 in the Minkowski limit is not a separate postulate: it is the flat-space restriction of the diffeomorphism invariance derived in Chapter 4. The projected field theory is automatically relativistic for the same reason the projected gravity is generally covariant.

Two distinct discrete structures arise once the field theory exists, and they must not be conflated. The first is topological. The vacuum manifold $\mathcal{V} \cong S^1$ has fundamental group $\pi_1(S^1) = \mathbb{Z}$, so the phase θ can wind around $\mathcal{V}$ only an integer number of times: a configuration cannot return to itself across a fractional winding without discontinuity. This integer is the U(1) charge, and it is the origin of the discrete charge spectrum $L = n\hbar$ derived in the vacuum-manifold discussion above, angular momentum in the far-threshold regime, spin near it. This quantisation is genuinely structural: it follows from the topology of $\mathcal{V}$ alone, with no quantisation postulate imposed.

The second discrete structure is the Fock space, and it is a separate matter. The occupation number of field quanta is the eigenvalue of $N = a^\dagger a$, an integer unrelated to the topological winding number: the former counts excitations of the projected field, the latter labels superselection sectors of the phase. The Fock space arises in PST exactly as in ordinary field theory, by canonical quantisation of the projected Lorentzian action S_2 once it exists, that is, once sublimation has supplied the timelike direction along which equal-time commutation relations can be posed. PST therefore derives the arena in which canonical quantisation takes place and the charge spectrum the resulting states must carry. The canonical commutation relations $[\hat{x}_\mu, \hat{p}_\nu] = i\hbar\,\delta_{\mu\nu}$ are derived in §9.4 from the Boolean gradient; the Born rule is derived in §9.2.

The existence of the fermionic sector follows structurally from the Boolean-CAR isomorphism (§8.15); the functional-analytic treatment of the fermionic Fock space is closed at proof-detail level via the three closures of §8.16 (CAR-to-continuum Mosco convergence, Wightman axioms verification, substrate-derived Dirac equation), with the spin-statistics rule as a structural theorem from Boolean exclusion of P1. The narrower content of the present construction is: the projected field theory is a standard Lorentzian field theory carrying a topologically quantised U(1) charge, and its excitation spectrum is discrete for the ordinary reason, not because winding number and occupation number coincide.

The projection operator Π carries the intrinsic scale d_0 derived in Chapter 10. Below d_0 the continuum description of Π fails; above it the smooth field theory holds. The scale $1/d_0$ is a physical ultraviolet cutoff, not a calculational regulator. The divergences of standard QFT, recovered as $d_0 \to 0$, are artefacts of taking the continuum limit of a projection that was never continuous at arbitrarily short distances; renormalisability is the statement that low-energy observables are insensitive to d_0, which holds for separations $\gg 7$ nm precisely as the Casimir analysis of Chapter 10 requires.

The status of three items not directly delivered by this projected-field-theory construction is summarised explicitly (each is either derived in a later section or has a residual open piece, stated below):

The Born rule. The identification of $|\psi|^2$ near threshold with measurement probability is derived from the Bernoulli measure and Gleason's theorem applied to $L^2(\mathcal{P}(D), \mu)$ in §9.2. The full measurement postulate (collapse rule) is closed at proof-detail level via the three closures of §9.3 (explicit von Neumann measurement Hamiltonian, decoherence inheritance, threshold-crossing uniqueness + irreversibility theorem). The fermionic-sector treatment is closed at proof-detail level via §8.16 (three closures: CAR-to-continuum convergence, Wightman axioms, substrate-derived Dirac equation).

Spin and statistics. The excitations constructed here are bosonic, carrying integer winding charge. The fermionic sector and the spin-statistics connection are not obtainable from the single complex order parameter alone; they are derived separately in Chapter 8 from the CAR algebra on the substrate's Boolean distinction primitive, where the half-integer-spin excitations inherit the correct anticommutation relations as a structural consequence.

The full interacting theory. The free-field Fock space and the ϕ^4-type vertices are exhibited; a rigorous treatment of the interacting vacuum and its renormalisation is deferred.

The relationship between PST and quantum field theory is now exactly parallel to its relationship with General Relativity, and the parallel is the point. QFT is neither a competitor to PST nor identical to it. It is the Lorentzian projection, through Π, of the fluctuation spectrum of $\mathcal{F}$ about $\mathcal{V}$, just as General Relativity is the projection of the vacuum value of the same functional. PST is not itself a quantum field theory and could not be: $\mathcal{F}$ is Euclidean, atemporal, and

precausal, and a quantum field theory requires the Lorentzian action S that exists only after the threshold is crossed. To demand that PST be a QFT at the foundational level would be to demand that it presuppose the spacetime, causal ordering, and time-ordered structure it sets out to derive. That QFT has no precausal counterpart is, as with the absence of a precausal action S, a theorem of the framework rather than a shortcoming.

7.21. Interacting vacuum and d_0-regulated renormalisation

The substrate grain d_0 imposes a physical momentum cutoff $M_* = \hbar c/d_0$ above which no field modes exist. This is not a Wilsonian regulator introduced for convenience and to be removed; it is a structural property of the projection Π. All loop integrals in the projected field theory are therefore UV-finite at every order, with the hard cutoff M_* replacing the formal $\Lambda \to \infty$ limit of standard QFT.

One-loop Higgs mass correction. Write the order parameter in unitary gauge as $\psi = (r_0 + h/\sqrt{2})\, e^{i\theta}$, where h is the amplitude fluctuation (PST Higgs) of tree-level mass m_h and θ is the massless Goldstone mode. At one loop and momentum cutoff M_*, the quadratically sensitive correction to the Higgs mass squared receives contributions from the top quark, electroweak gauge bosons, and Higgs self-coupling [94]:

$$\delta m_h^2 \;=\; \frac{M_*^2}{16\pi^2}\Big[\underbrace{6\lambda + \tfrac{9}{4}g^2 + \tfrac{3}{4}g'^2}_{\text{bosonic }(+)} \;\underbrace{-\,6\,y_t^2}_{\text{top }(-)}\Big], \tag{103}$$

the standard one-loop quadratic-sensitivity (Veltman) bracket, where λ, g, g', y_t are the Higgs self-coupling (doublet convention), $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ gauge couplings, and the top Yukawa. Substituting Standard Model values at the electroweak scale [93] ($\lambda \approx 0.13$, $g \approx 0.65$, $g' \approx 0.36$, $y_t \approx 0.94$):

$$\delta m_h^2 \;\approx\; \frac{0.78 + 0.95 + 0.10 - 5.30}{16\pi^2}\, M_*^2 \;\approx\; -\frac{3.47}{16\pi^2}\, M_*^2 \;\approx\; -0.022\, M_*^2. \tag{104}$$

This is the *full* one-loop sensitivity, dominated by the top loop and *negative*. It is the physical, coupling-weighted quantity, and it is *not* the same object as the scalar-sector self-loop $+(3/2)M_*^2/(16\pi^2)$ used below to extract M_*: the latter keeps only the Higgs self-coupling piece, the former includes top, W, and Z. The substrate eigenmode cancellation does not reach the top/W/Z terms (see the scope discussion below), so equation (104) stands as the honest physical sensitivity.

Naturalness. For $M_* = 4\pi m_h\sqrt{2/3} \approx 1.285$ TeV:

$$|\delta m_h^2| \;\approx\; 0.022 \times (1285\ \mathrm{GeV})^2 \;\approx\; (190\ \mathrm{GeV})^2. \tag{105}$$

The fine-tuning parameter [95] is $\Delta \equiv |\delta m_h^2|/m_h^2 \approx (190)^2/(125)^2 \approx 2.3$, well within the

naturalness bound $\Delta \lesssim 10$. For comparison, with a Planck-scale cutoff $M_P \approx 1.22 \times 10^{19}$ GeV, the same formula gives $\Delta \approx 0.022\,(M_P/m_h)^2 \approx 2.1 \times 10^{32}$, the standard hierarchy problem. The PST physical cutoff eliminates this: the bare Higgs mass at the M_* scale, $m_{h,0} \approx \sqrt{m_h^2 + |\delta m_h^2|} \approx 228$ GeV, is of order M_*, not M_P. The hierarchy problem is absent from the d_0-regulated theory by construction, not resolved by a mechanism.

Vacuum stability. The sign of the effective μ^2 term determines whether spontaneous symmetry breaking persists. The one-loop correction to μ^2 (the negative mass-squared driving SSB) is $\delta\mu^2 = -\delta m_h^2/2 \approx +0.011\,M_*^2 > 0$: the quantum correction deepens the sombrero potential rather than flattening it. The renormalised vacuum amplitude is $r_0' = v'/\sqrt{2}$ where $v'^2 = v^2\,(1 + |\delta m_h^2|/m_h^2) \approx 3.3\,v^2$, shifting the electroweak scale from 246 GeV to ≈ 448 GeV for fixed bare parameters. The minimum exists and is stable; its precise location is set by renormalisation conditions that fix $v_{\rm phys} = 246$ GeV. Spontaneous symmetry breaking persists at one loop in the d_0-regulated theory.

Wilsonian effective field theory. Integrating out all PST modes with momenta $\mu < |k| < M_*$ generates the Wilsonian effective action at scale μ [96]:

$$S_{\rm eff}[\phi\,;\mu] \;=\; S_{\rm SM+GR}[\phi] \;+\; \sum_{n\geq 5} \frac{c_n^{(i)}}{M_*^{n-4}}\,\mathcal{O}_n^{(i)}[\phi], \tag{106}$$

where $\mathcal{O}_n^{(i)}$ are dimension-n operators and $c_n^{(i)} = O(1)$ are Wilson coefficients. At leading order the effective theory is precisely the Standard Model coupled to gravity. The first correction enters at dimension 6, suppressed as $(E/M_*)^2$; at $E = 1$ GeV this is $(10^{-3})^2 = 10^{-6}$, consistent with all precision tests.

Renormalisation group. Below M_* no PST modes propagate; the β-functions for all couplings are exactly those of the Standard Model (no Kaluza-Klein tower, no threshold corrections above M_*). Running from M_* down to the electroweak scale follows the Standard Model RGEs without modification.

Vacuum energy. At one loop, the zero-point energy density in the M_*-cutoff scheme is $\rho_\Lambda^{(1)} \sim M_*^4/(16\pi^2) \approx (362\ \text{GeV})^4$, a factor $M_P^4/M_*^4 \approx 8.1 \times 10^{63}$ smaller than the naive QFT prediction. The observed cosmological constant $\Lambda_{\rm obs} \approx (10^{-3}\ \text{eV})^4$ is still far below this one-loop estimate; the remaining gap is the standard Standard-Model cosmological-constant fine-tuning problem, identified as inherited rather than solved in §7.15 (joint analysis of Mechanism A and Mechanism B).

The vacuum-stability question is thereby resolved at leading loop order: the d_0-regulated interacting vacuum is UV-finite, stable, and reproduces the Standard Model as its low-energy EFT.

One-loop relation $M_* = 4\pi m_h\sqrt{2/3} \approx 1285$ GeV (parameter-free derivation under the corrected

O(4) *one-loop count, Comp 109; the earlier off-by-one of Computations 67, 69 is closed by the substrate-vs-emergent distinction of Computations 93, 97 – see §7.21).* Two inputs enter exactly rather than as free parameters. First, the PST boundary condition: the LG modal potential at the threshold is $F[\psi, 0] = \frac{1}{4}\psi^4$ (equation (23)), so the tree-level Higgs mass term vanishes exactly at M_* (the coefficient $a_{\rm LG}(M_*) = 0$ is not a tuning condition but the definition of the threshold); the physical Higgs mass is therefore entirely radiative. Second, the quartic coupling is the universal Landau-Ginzburg coefficient [8]: $\lambda_{\rm PST}(M_*) = \frac{1}{4}$.

At the scale M_* electroweak symmetry is unbroken ($M_* \gg v$), so all four real components of the complex doublet propagate as an O(4) multiplet with common quartic $\lambda_{\rm PST}$. The O(N) scalar vertex $V_{ijkl} = 2\lambda(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ contracted into the one-loop seagull $(1/2)\, V_{ijkl}\, \delta_{kl}$ gives a coefficient $(N+2)\,\lambda$ in front of $M_*^2/(16\pi^2)$, with all three index contractions ($\delta_{ij}\delta_{kl} \to N\,\delta_{ij}$ and the two same-component contractions $\to 2\,\delta_{ij}$) enumerated explicitly in Comp 109. For $N = 4$ this gives the standard O(4) result

$$\delta m_h^2\big|_\Phi \;=\; \frac{(N+2)\,\lambda_{\rm PST}}{16\pi^2}\, M_*^2 \;=\; \frac{6 \times \frac{1}{4}}{16\pi^2}\, M_*^2 \;=\; \frac{3}{2}\,\frac{M_*^2}{16\pi^2}, \tag{107}$$

with broken-phase cross-check 3λ (physical Higgs $hhhh$ loop) $+3\lambda$ (three Goldstones) $= 6\lambda$ agreeing with the symmetric-phase O(4) result. The Higgs self-loop coefficient is therefore $C_H \equiv (N+2)\,\lambda_{\rm PST} = 6\lambda_{\rm PST} = 3/2$ exactly. The cancellation of the PST scalar eigenmode sector follows from the spectral decomposition of the PST modal field in the eigenbasis of the Boolean Laplacian L^B (§7.6, equation (46)). Every non-zero eigenmode ϕ_k (labelled by a non-empty subset $S \subseteq [D]$, with mass $m_k = |S|^{1/2} M_* \geq M_*$) has field-dependent mass $m_k^2(\phi) = |\phi|^2$ from the LG quartic (equation (23)), giving a universal coupling coefficient $c_k = 1$ for all $k \neq 0$. The Bernoulli measure (equation (8)) assigns equal weight to bosonic ($|S|$ even) and fermionic ($|S|$ odd) eigenmodes, with $N_B = N_F = 2^{D-1}$ modes in each sector (§8.15):

$$\delta m_h^2\big|_{\rm PST\,scalar} \;=\; \frac{M_*^2}{16\pi^2}\,\big[N_B - N_F\big] \;=\; 0. \tag{108}$$

The Higgs zero mode ($S = \emptyset$, eigenvalue 0 of L^B) is excluded because it is the order parameter. The SM fermionic and gauge contributions (top quark, W, Z) arise from distinct PST sectors (the fermionic occupation-number sector (§8.15) and the gauge sector of the internal algebra (Chapter 8)) and cancel by the equal-weighting postulate applied to those sectors, the structural component of which is established by the kernel normalisation derived in §7.9. With equation (108) and $C_H = 3/2$, equation (107) is the complete one-loop result *for the PST scalar sector*, predicting the modal scale from the measured Higgs mass:

$$M_*^2 \;=\; \frac{16\pi^2\, m_h^2}{C_H} \;=\; \frac{32\pi^2}{3}\, m_h^2, \qquad M_* \;=\; 4\pi\, m_h \sqrt{\tfrac{2}{3}} \;\approx\; 1285\ \text{GeV}, \tag{109}$$

using $m_h = 125.25$ GeV [93] as input. The denominator $16\pi^2$ is the one-loop factor; the

$\sqrt{2/3}$ rescaling relative to the older $4\pi m_h$ form comes from the O(4) vertex contraction $(N+2)\lambda = 6\lambda$ rather than the naive $N\lambda = 4\lambda$ (Comp 109). The two-loop correction *from the Higgs self-interaction piece*, estimated within the framework of reference [94], is $\delta m_h/m_h = \lambda_{\mathrm{PST}}/(32\pi^2) \approx 7.92 \times 10^{-4}$, a 0.08% shift, well below the experimental uncertainty on m_h and negligible for that piece of the prediction. Stability of the full result through two loops additionally requires that the top-quark and gauge two-loop contributions also cancel; the one-loop scalar-sector argument above does not by itself establish this.

Scope of the cancellation argument. The M_* relation is a *scalar-sector* result: it follows from the Higgs self-loop alone, and a clean separation between what equation (108) establishes and what would be needed to extend it to the full Standard Model is essential to stating the status honestly. An explicit one-loop diagrammatic accounting separates the two:

1. *The PST scalar eigenmode sector cancellation (eq.* (108)*) is rigorous.* Three ingredients are required, all structural:

   - *Equal counts.* $N_B = N_F = 2^{D-1}$ follows from the binomial identity $\sum_{k\,\mathrm{even}} \binom{D}{k} = \sum_{k\,\mathrm{odd}} \binom{D}{k} = 2^{D-1}$; combinatorially exact.
   - *Universal coupling.* $C_H = (N+2)\lambda_{\mathrm{PST}} = 6\lambda_{\mathrm{PST}} = 3/2$ from the single LG quartic and the O(4) one-loop vertex contraction (Comp 109); every non-zero eigenmode of L^B inherits the same field-dependent mass $m_k^2(\phi) = |\phi|^2$ at this level, giving the same one-loop coefficient $c_k = 1$ for all $k \neq 0$. Universal at the substrate level by construction (one quartic, one coefficient).
   - *Sign assignment.* The opposite loop signs of bosonic and fermionic modes follow from the CAR-algebra realisation of P1's Boolean configurations (§8.15): even-$|S|$ Boolean configurations realise commuting (bosonic) statistics and odd-$|S|$ configurations realise anticommuting (fermionic) statistics, with the standard spin-statistics correspondence carrying through to the one-loop $(-1)^F$ factor. Algebraic, not asserted.

   With all three ingredients structural, the $\delta m_h^2|_{\mathrm{PST\,scalar}} = 0$ result is internally rigorous.

2. *The SM top/W/Z extension is* ***not*** *established; it is an open problem.* The substrate eigenmode cancellation of equation (108) operates on the substrate Walsh modes, which (Comp 105) populate the UV shell between M_* and the matched cutoff $\Lambda = \sqrt{D}$. The physical top, W, Z, and Higgs loops that determine the observed δm_h^2 run in the *emergent* EFT shell below M_*, where (Proposition F4) the theory is exactly the Standard Model with its measured, non-universal couplings. By the very disjointness that Comp 105 establishes to defuse double counting, the substrate-layer cancellation cannot reach into the EFT-layer loops; it does not delete the top/W/Z contributions to the physical Higgs mass. Indeed the empirical one-loop Veltman quadratic-sensitivity bracket at M_* is large

and *negative*, $\mathcal{V}(M_*) \approx -3$ (§7.21), dominated by the top loop — not the $+m_h^2$ that a clean cancellation would leave. An earlier version of this paper (Comps 107, 108) argued that "sector-blindness" of the substrate binomial identity transfers the cancellation to the SM sectors; that argument is *withdrawn*. Sector-blindness holds for the substrate count, but it does not control the coupling-weighted EFT-layer quantity, and the universality premise it rests on (every Walsh mode carrying $c_S = 1$ at M_*) is contradicted by the observed Yukawa hierarchy, which §8.20 itself assigns to $T(C)$-contingency rather than to RGE running of a universal value. Whether an explicit cancellation mechanism for the EFT-layer top/W/Z quadratic sensitivities exists — and what partner spectrum or matching structure it would require — is an open question (open-research item 1.3), not a closed result.

The honest status is therefore: $m_h^2 = (3/2)\, M_*^2/(16\pi^2)$ is a *scalar-sector* one-loop result — a parameter-free relation between the Higgs mass and the modal scale for the Higgs self-loop in isolation — and $M_* = 4\pi m_h\sqrt{2/3} \approx 1.29$ TeV is the modal scale it implies. This rests on the substrate-vs-emergent zero-mode distinction of Computations 93 and 97: the original boson–fermion cancellation argument carried an apparent zero-mode off-by-one (Computation 67, Appendix C) under the implicit reading that the substrate-side Boolean zero mode $S = \emptyset$ IS the emergent SM Higgs doublet. Comps 93 and 97 make the distinction explicit via the projection chain $\psi \to \Pi(\psi) \to H_{\text{doublet}}$: the substrate Walsh zero mode $\chi_\emptyset$ maps under Π to the constant function on $M = \mathbb{R} \times S^3$ (vacuum direction), structurally distinct from the emergent 4-component Higgs doublet generated by inner fluctuations of the Dirac operator on $M \times F$. Under this distinction, including the zero mode in N_B does not double-count the emergent doublet (captured by $C_H = 6\lambda = 3/2$), and the eigenmode-sum identity $N_B - N_F = 0$ combined with $C_H = 3/2$ delivers $\delta m_h^2\big|_{\text{scalar}} = (3/2)\, M_*^2/(16\pi^2)$. The result is *conditional* on the EFT-layer top/W/Z quadratic sensitivities being separately accounted for — which, as item 2 above states, is not done here and is in tension with the computed Veltman residual.

Computational disclosure. Computation 67 isolates the off-by-one combinatorially; Computation 69 walks each of the three proposed rescues in detail and shows why each fails: (a) the eaten-Goldstone rescue mixes symmetric-phase and broken-phase mode counts at different RG scales without an explicit matching argument; (b) putting the zero mode back into the cancellation sum gives $N_B - N_F = 0$ but no leftover $+M_*^2/(16\pi^2)$ for the Higgs self-mass, yielding $m_h = 0$ at one loop; (c) complementation $S \to S^c$ pairs even/odd parities only for odd D, and even then the zero-mode/full-set pair is broken by the exclusion of the zero mode as the order parameter, leaving a -1 deficit. The combined accounting of Computations 67 and 69 is the present grounds for the NDA-only framing in earlier drafts. Computations 93 and 97 close the off-by-one via the substrate-vs-emergent distinction in the PST projection chain. Comp 93 identifies the fourth-rescue candidate: the substrate-side Boolean zero mode $S = \emptyset$ is the substrate's constant-mode profile, distinct from the emergent post-projection

4-component Higgs doublet (captured separately by $C_H = 6\lambda_{\rm PST} = 3/2$ in equation (107)). Comp 97 validates this distinction via explicit projection-chain analysis: the substrate Walsh zero mode $\chi_\emptyset$ maps under Π to the constant function on $M = \mathbb{R} \times S^3$ (vacuum direction), while the emergent SM Higgs doublet (4 real components H^+, H^0) arises from inner fluctuations of the Dirac operator on $M \times F$, generated by the $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ factor. These are structurally distinct objects at different layers of the projection chain $\psi \to \Pi(\psi) \to H_{\rm doublet}$. Comp 105 verifies this non-double-counting at the diagram level: the substrate $\chi_\emptyset$ loop (between $\Lambda = \sqrt{D}$ and M_*) and the emergent C_H loop (between M_* and the IR) integrate over disjoint momentum support meeting only at the single matching scale M_* (measure zero); the substrate full-trace partition function decomposes exactly as $Z^{\rm full}_{\rm sub}(D) = Z^{\rm IR}_{\rm sub}(D, k_*) + Z^{\rm UV}_{\rm sub}(D, k_*)$ across the M_*-shell partition with machine-zero residual. Including the zero mode in N_B therefore does not double-count the emergent doublet, and the total scalar-sector contribution $C_H + (N_B^{\rm full} - N_F) = 3/2 + 0 = 3/2$ gives $\delta m_h^2\big|_{\rm scalar} = (3/2)\, M_*^2/(16\pi^2)$, i.e. $m_h = M_*\sqrt{6}/(8\pi)$ for the Higgs self-loop in isolation.

The PST scalar sector therefore predicts, *of itself*, a definite scalar self-mass relation; the extension to the full Standard Model is open (item 2 above). If a full-SM cancellation of the top/W/Z quadratic sensitivities were to hold, it would require Boolean partner states near $M_* \approx 1.29$ TeV with Higgs couplings matched to their SM counterparts; such partners would be subject to direct LHC searches (vector-like quarks and top partners are currently excluded to $\gtrsim 1.4$ TeV) and would conflict with the "no new light states" premise of the SMEFT analysis in §10.9. Since the full-SM cancellation is not established, no such partner spectrum is claimed here; the question of whether it exists, and how it would evade the searches, is part of open-research item 1.3. The Casimir d^{-6} correction of Chapter 10 sets in at the Landau-Ginzburg coherence scale d_0; its power-law exponent is a parameter-free structural consequence of substrate discreteness, while the onset scale d_0 is set by the LG dynamics.

Field normalisation and the Higgs VEV. The PST quartic coefficient $b(M_*) = 1/4$ is fixed by the LG modal potential; it is not a running coupling. The SM quartic $\lambda_{\rm SM}$, by contrast, runs from its electroweak-scale value $\lambda_{\rm SM}(v) = m_h^2/(2v^2) \approx 0.1294$ upward to $M_* \approx 1285$ GeV via the one-loop SM beta function [93]:

$$\beta_\lambda \;=\; \frac{1}{16\pi^2}\big[24\lambda^2 + 12\lambda y_t^2 - 6y_t^4 - 3\lambda(3g_2^2 + g_1^2) + \tfrac{3}{8}(2g_2^4 + (g_1^2 + g_2^2)^2)\big], \tag{110}$$

integrated numerically from $\mu = m_t = 173$ GeV to $M_* \approx 1285$ GeV with the full coupled system $\{\lambda, y_t, g_2, g_1, g_s\}$, giving $\lambda_{\rm SM}(M_*) \approx 0.0927$. These two couplings are defined in different field normalisations; their ratio is the *field normalisation factor*:

$$Z^2 \;\equiv\; \frac{\lambda_{\rm SM}(M_*)}{b(M_*)} \;=\; \frac{0.0927}{0.25} \;=\; 0.371. \tag{111}$$

Numerically $e^{-1} = 0.36788\ldots$, so $Z^2/e^{-1} = 1.008$, a 0.8% deviation, well within the one-loop uncertainty of approximately 10% from two-loop omissions. No parameter was adjusted to produce this; it follows entirely from PST fixing $b(M_*) = 1/4$, from $M_* = 4\pi m_h\sqrt{2/3}$, and from the observed (m_h, v).

The proximity of $Z^2 = 0.371$ to $e^{-1} = 0.36788\ldots$ is at 0.8% deviation at one loop. This subsection introduces it as an observed near-coincidence; its structural derivation – via the substrate's KO-tempered Bernoulli partition function bridge – is given in §7.22 below (Comps 84–92). Taking $Z^2 \approx e^{-1}$,

$$\lambda_{\rm SM}(M_*) \;=\; b(M_*)\cdot Z^2 \;\approx\; \frac{1}{4e} \;\approx\; 0.092. \tag{112}$$

Running $\lambda_{\rm SM}(M_*) = e^{-1}/4$ back down to the electroweak scale gives $\lambda_{\rm SM}(v) \approx 0.1294$, from which $v = m_h/\sqrt{2\lambda_{\rm SM}(v)} \approx 246.2$ GeV, 0.01% from the observed 246.22 GeV. *Conditional note:* this derivation of v uses the SM tree-level relation $m_h^2 = 2\lambda v^2$ plus the SM RGE running between v and M_*. The PST contribution is that the relation $Z^2 \approx e^{-1}$ (§7.21) pins $\lambda_{\rm SM}(M_*)$ without using v as an input. To the extent that $Z^2 \approx e^{-1}$ holds, v is determined by m_h and the SM RGE structure alone. The non-trivial claim is thus conditional: *if $Z^2 = e^{-1}$ holds exactly, then given the Standard-Model gauge structure and RGE running between v and M_*, the two electroweak observables (m_h, v) collapse to one, with v following from m_h.* Two caveats attach. First, the substrate-side factor of $Z^2 = e^{-1}$ is derived structurally from P1–P3 in §7.23 below (via the tensor-product factorisation forced by P1's Bernoulli product measure, Comp 100; items (a) and (b) of the residual gap closed by Comp 98 via the mode-shell block-spin RG argument); the matching identification with the SM-side ratio is bridge premise (B), closed within PST in the substrate limit (§7.23). The 0.8% empirical agreement is at full-Buttazzo-level two-loop SM RGE running (and 5–7% at one-loop precision, Comp 91). Second, Z^2 is itself constructed from (m_h, v) — $\lambda_{\rm SM}(v) = m_h^2/2v^2$ already contains v, and the run-up to M_* inherits it — so the "v from m_h" map is only well-posed to the extent that the $v \to M_* \to v$ running is insensitive to the v being solved for; we assert but do not here demonstrate that self-consistency.

Candidate substrate-side identification. The observed value $Z^2 \approx e^{-1}$ has a structural counterpart on the substrate side. This subsection records the earlier *spectral-action* route to it, which rested on a single explicit conjecture; that route is superseded by the partition-function-level closure of §7.23, where the identification (bridge premise B) is derived within PST in the substrate limit by a wave-function-renormalisation argument and the unique-soft-mode theorem F11, so the conjecture stated here is no longer load-bearing. We give the spectral-action approach because it motivates that closure and supplies the substrate-side exactness it builds on. The full Chamseddine-Connes inner-fluctuation route $D + A + \varepsilon' JAJ^{-1}$ applied directly to the substrate spectral triple is examined in Comp 81 and shown to collapse at matched scaling, motivating the alternative partition-function-level identification developed in §7.22 and §7.23.

The substrate side is exact. Comp 62 establishes that at the matched scaling $\Lambda = \sqrt{D}$ of the substrate spectral triple, the Gaussian-cutoff substrate spectral action satisfies

$$\frac{S_{\text{sub}}}{2^D} \;=\; \frac{\operatorname{Tr} f(D_{\text{sub}}/\Lambda)}{\operatorname{Tr} 1} \;=\; f(1) \;=\; e^{-1}, \qquad f(x) = e^{-x^2}, \tag{113}$$

D-independently and exactly, because the substrate Dirac eigenvalues $\pm\sqrt{D}$ each with multiplicity 2^{D-1} collapse to ± 1 under the matched scaling. All 2^D eigenmodes sit at the cutoff edge with weight $f(1) = e^{-1}$, so the trace ratio is $f(1) = e^{-1}$. This is the substrate analogue of the Chamseddine-Connes spectral-action moment $f_0 = f(0)$ for the continuum Connes-Dirac triple [49, 48].

The candidate identification. We conjecture:

$$Z^2 \;\equiv\; \frac{\lambda_{\text{SM}}(M_*)}{b(M_*)} \;\stackrel{?}{=}\; \frac{S_{\text{sub}}}{2^D} \;=\; e^{-1} \qquad \text{(exact, } D\text{-independent)}. \tag{C$_{Z^2}$}$$

Under this identification, $Z^2 = e^{-1}$ is structurally exact, and the 0.8% observed deviation lives entirely on the SM-RGE side as the two-loop truncation error of the running $\lambda_{\text{SM}}(v) \to \lambda_{\text{SM}}(M_*)$. Empirically the two-loop SM beta functions shift $\lambda_{\text{SM}}(M_*)$ at the few-percent level [51], in the direction predicted by the identification.

The structural conjecture. Identification (C$_{Z^2}$) rests on:

Conjecture (spectral-action Higgs-quartic identity). *In the Chamseddine-Connes spectral-action expansion of the PST discrete substrate triple at the matched scaling $\Lambda = \sqrt{D}$, the coefficient of the Higgs-quartic term $\lambda|H|^4$ in the projected effective action equals $b(M_*)\cdot S_{\text{sub}}/2^D$, with no additional dimensionless one-loop factor.*

The conjecture is a finite, structural calculation in the spectral-action framework: it asks whether the Higgs-quartic coefficient extracted from $S(D, \Lambda, f) = \operatorname{Tr} f(D/\Lambda)$ for the PST substrate triple at matched scaling is exactly $b(M_*) \cdot S_{\text{sub}}/2^D$. In the *continuum* Connes-Chamseddine-Marcolli triple, the analogous coefficient is fixed by the moments f_0, f_2, f_4 of the cutoff function [50]; the conjecture's content is that for the *discrete* substrate triple, where the entire spectrum is concentrated at the cutoff edge, the moment structure collapses to the trace ratio $f(1) = e^{-1}$ at one-loop order.

The remaining gap. Comp 71 verifies the substrate side exactly, computes the SM side at one loop, and confirms the quantitative consistency. Comp 72 carries out the explicit Taylor expansion of the substrate spectral action under a Yukawa-like Higgs perturbation $D_\phi = D_{\text{sub}} + y\,\phi\,\Sigma_X$, extracting the ϕ^4 coefficient symbolically. The result at matched scaling

is

$$\left.\frac{S(D_\phi,\Lambda,f)}{2^D}\right|_{\phi^4} \;=\; e^{-1}\cdot\frac{y^4}{2\,\Lambda^4} \;=\; e^{-1}\cdot\frac{y^4}{2D^2}. \tag{114}$$

For (C_{Z^2}) to deliver the SM Higgs quartic $\lambda_H = b(M_*)\cdot e^{-1} = (1/4)\,e^{-1}$, the substrate-Higgs Yukawa coupling must satisfy

$$\frac{y^4}{2\,\Lambda^4} \;=\; \frac{1}{4} \qquad\Longleftrightarrow\qquad y \;=\; \frac{\Lambda}{2^{1/4}} \;\approx\; 0.8409\,\Lambda. \tag{115}$$

The conjecture is therefore reduced from "the spectral-action Higgs-quartic identity" to the specific single-scalar condition (115): the substrate-Higgs Yukawa coupling, derived structurally from a Chamseddine-Connes inner fluctuation of D_{sub} by elements of the internal algebra $A_F = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$ at matched scaling, must equal $\Lambda/2^{1/4}$. This is a finite calculation in the spectral-triple framework [50], not a new postulate. Under its resolution, $Z^2 = e^{-1}$ moves from numerical near-coincidence to structural theorem and the "v from m_h" map (above) inherits structural rather than empirical status.

The Yukawa calibration as a 4th-root of the Bernoulli measure. The required Yukawa (115) admits a more transparent reading. Substituting $b = 1/4$ (the LG quartic of postulate P3) into (115) gives the general form

$$\frac{y_{\mathrm{substrate}}}{\Lambda} \;=\; (2b)^{1/4} \;=\; \mu_{\mathrm{site}}^{1/4}, \qquad \mu_{\mathrm{site}} := 2b = \tfrac{1}{2}, \tag{116}$$

i.e. the substrate-Higgs Yukawa coupling at matched scaling is the 4th root of the per-site Bernoulli probability of the substrate measure. This is not a coincidence: $\mu_{\mathrm{site}} = \frac{1}{2}$ is the per-site Bernoulli probability of $\mu = \bigotimes_{a\in D}\mathrm{Bern}(\frac{1}{2})$ delivered structurally by P1 (distinguishability; §1.7), and the 4th-root power matches the dimension of the emergent Lorentzian spacetime $M = \mathbb{R}\times S^3$ at the matched scaling. Equation (116) therefore expresses the Z^2 conjecture as a single calibration linking three postulates:

P1 (distinguishability) supplies $\mu_{\mathrm{site}} = 1/2$.

P3 (LG modal potential) supplies $b = 1/4$.

Spacetime emergence supplies the 4th-root power.

The open content of the Z^2 conjecture is then: does the Chamseddine-Connes inner-fluctuation calculation deliver $y_{\mathrm{substrate}} = \Lambda\cdot\mu_{\mathrm{site}}^{1/4}$ *exactly*, as the unique self-consistent Yukawa at matched scaling on the discrete substrate triple? Each ingredient (μ_{site}, the 4th-root, the matched-scaling normalisation) has independent structural origin; the conjecture is that they combine as (116).

First-cut inner-fluctuation calculation (Computation 74). A first-cut attempt at the calculation, with the simplest natural single-generator inner fluctuation $A = i\phi\,[D_{\mathrm{sub}},\omega]/2$ where $\omega = e_1e_2$ is the quaternion volume element in $\mathrm{Cl}(0,D)$, gives an explicitly computable perturbed Dirac

$D_\phi^2 = (D + 2\phi^2)\, I$ and a ϕ^4 coefficient in $S/2^D$ of $2e^{-1}/D^2$. Matched against equation (114), this delivers $y = \sqrt{2}$ (D-independent in absolute terms) and therefore $y/\Lambda = \sqrt{2/D}$, which *decreases* with D. The Comp 72 target $y/\Lambda = \mu_{\text{site}}^{1/4} = 2^{-1/4}$ is D-*independent*. The two are incompatible: the naive single-generator inner-fluctuation does not close (116).

Computation 74 documents three possible resolutions:

1. $Z^2 \approx e^{-1}$ is a genuine 0.8% numerical near-coincidence on the SM side; the substrate-side $S_{\text{sub}}/2^D = e^{-1}$ remains exact (Computation 62) independently of any Yukawa identification. The conservative reading: no structural identification at the Yukawa level survives.

2. The Higgs perturbation should be parameterised over multiple A_F generators (or via the full Chamseddine-Connes inner fluctuation $D \to D + A + \varepsilon' JAJ^{-1}$ with order-one and Reality constraints), not just a single ω. This is a multi-step CC-machinery calculation that has not been carried out here.

3. The matched scaling $\Lambda = \sqrt{D}$ is not the right cutoff identification; a different $\Lambda(D)$ prescription would deliver $y/\Lambda = 2^{-1/4}$ from the same perturbation.

The honest reading after Computation 74 is therefore: (C_{Z^2}) remains *candidate* not *closed* at the Chamseddine-Connes inner-fluctuation level; the simplest natural attempt does not deliver the required Yukawa. Subsequent work has demonstrated that this is a *structural obstacle*, not a calculational shortfall: see the following subsection.

The CC inner-fluctuation bridge is structurally obstructed. Comps 84–86 establish that the obstacle to closing (C_{Z^2}) via Chamseddine-Connes inner fluctuations is not calculational but structural. Comp 84 extends the single-generator Comp 74 calculation to multi-generator and Bernoulli- activated families; neither delivers the target $y/\Lambda = 2^{-1/4}$ at a structurally natural value of K (the integer count fails) or p (the Bernoulli activation probability fails). Comp 85 reduces the identification to the Connes-Chamseddine-Marcolli moment-extraction level and identifies the same obstacle. Comp 86 makes the obstacle explicit: the Seeley-DeWitt heat-kernel asymptotic expansion that delivers λ from the cutoff moments in continuum CCM presupposes a continuous spectrum. For the discrete substrate triple at matched scaling, all eigenvalues sit at $\pm\sqrt{D} = \pm\Lambda$ exactly, so the heat-kernel expansion collapses ($a_n = 0$ for all $n \geq 1$). The CCM machinery cannot deliver the Higgs-quartic coefficient from the discrete substrate triple. This rules out option 2 (more sophisticated CC inner fluctuation) as a path to closure; the bridge must come from a fundamentally different mechanism.

7.22. Closure via the KO-tempered Bernoulli partition function

With the CC inner-fluctuation bridge structurally obstructed (§ above), an alternative substrate-side derivation of e^{-1} is available via the Bernoulli moment-generating function – and turns

out to admit a clean Wilsonian matching to the SM-side Z^2.

An independent substrate-side derivation of e^{-1} (Comp 88). The substrate Bernoulli measure $\mu = \bigotimes_{a\in D} \mathrm{Bern}(\frac{1}{2})$ has empirical mean $\bar{X}(C) := |C|/D$ with asymptotic moment-generating function

$$\mathbb{E}_\mu[\exp(c\,\bar{X})] \;=\; \big((1+\mathrm{e}^{c/D})/2\big)^D \;\xrightarrow{D\to\infty}\; \exp(c\,\mu_{\mathrm{site}}) \;=\; \exp(c/2), \tag{117}$$

delivering e^{-1} at the structurally-forced scale

$$c \;=\; -2, \tag{118}$$

where the "2" is the spectral range of a single binary distinction, $2 = 1-(-1)$: the exchange at each substrate site is a self-adjoint involution with eigenvalues $\{+1,-1\}$, so its range is fixed at 2 with no freedom (P1's minimal property distinction is binary; this is the one-bit Clifford $\mathrm{Cl}(1,0)$ range under an equivalent name). The numerically identical $\mathrm{KO}_{\mathrm{total}} \bmod 8 = 2$ from Bott periodicity shares the value but is a gauge- and spacetime-sector fact, never an input here; it is not the load-bearing route. All three ingredients ($\mu_{\mathrm{site}} = 1/2$ from P1, the binary-distinction gap 2 from P1, asymptotic limit from Cramér-Bernoulli machinery [148]) are structurally derived.

The value e^{-1} is *robust across the cutoff-function family* $f(x) = e^{-x^p}$ at the matched edge $x = 1$: any exponential-of-power cutoff gives $f(1) = \mathrm{e}^{-1}$ at the matched scaling, so the substrate-side value is the same number under the Gaussian, exponential, or any $p \in (0,\infty)$ choice. The Bernoulli-MGF route asymptotically delivers $\mathbb{E}_\mu[\exp(-2\bar{X})] \to \mathrm{e}^{-1}$ via the $f(x) = e^{-x}$ form (Comp 100, exponential cutoff forced by P1's bit-independence); the spectral-action route via $f(x) = e^{-x^2}$ at $x = 1$ also gives e^{-1} (Comp 62, Gaussian). These routes use different operators (spread Boolean Laplacian eigenvalues $2|S|$ in Comp 100; degenerate-spectrum collapse $\pm\sqrt{D}$ in Comp 62), and the Bernoulli-MGF route is asymptotic in D (e.g. at $D = 6$, Comp 100 gives 0.40, $\sim 8.6\%$ high; at $D = 10$, 0.39, $\sim 5\%$ high; at $D = 100$, 0.3697, $\sim 0.5\%$ high; at $D = 10^4$, deviation $\sim 5\times 10^{-5}$) rather than D-independent. The two routes give the same asymptotic number from structurally distinct expressions.

The substrate Higgs Hamiltonian (Comp 89). For the Bernoulli-MGF identification to function as a substrate-to-SM bridge, the substrate Higgs Hamiltonian must be identified structurally. Comp 89 derives $H_{\mathrm{Higgs}}(C)$ from three constraints:

(i) *Additivity.* The Bernoulli measure factors across independent sites; an extensive substrate Hamiltonian must be additive, $H(C) = \sum_a h_a(B_a)$.

(ii) *Matched scaling.* At $\Lambda = \sqrt{D}$, per-site energy is rescaled by $1/\Lambda^2 = 1/D$.

(iii) *Uniformity.* P1 distinguishability treats all sites equivalently; per-site energy is uniform

across a.

Together, (i)–(iii) force

$$H_{\text{Higgs}}(C) \;=\; \frac{1}{D}\sum_{a\in D} B_a(C) \;=\; \bar{X}(C), \tag{119}$$

the empirical mean of bit-occupation, with *no free parameters*. The KO-tempered substrate partition function

$$Z_H(\beta_{\text{KO}}) \;:=\; \mathbb{E}_\mu[\exp(-\beta_{\text{KO}}\, H_{\text{Higgs}}(C))] \;\xrightarrow{D\to\infty}\; \mathrm{e}^{-1} \tag{120}$$

is therefore the substrate-side quantity corresponding to the SM-side Z^2, with $\beta_{\text{KO}} = 2$ the spectral range of a single binary distinction, $2 = 1 - (-1)$, forced by P1's binary alphabet (equivalently the one-bit Clifford range; the numerically equal $\text{KO}_{\text{total}} \bmod 8 = 2$ shares the value but is not the source).

The bridge identification: seven-step proof sketch (Comp 92). Combining the Wilsonian framework of §7.21 with the substrate-side derivations:

(1) The PST EFT below M_* is exactly the Standard Model (eq. (106); no PST modes propagate below M_*).

(2) The matching at M_* between the PST UV and the SM IR is given by the standard Wilsonian matching condition.

(3) The PST UV at M_* has bare coupling $\lambda_{\text{PST}}^{\text{bare}} = b = 1/4$ (LG modal potential, §1.5).

(4) The substrate UV at $\Lambda = \sqrt{D}$ contributes vacuum fluctuations between Λ and M_*, with measure μ.

(5) These fluctuations integrated over μ contribute a multiplicative factor $Z_H(\beta) = \mathbb{E}_\mu[\exp(-\beta\, H_{\text{Higgs}}(C))]$ to the bare coupling, with $H_{\text{Higgs}} = \bar{X}$ derived from substrate primitives (equation (119)).

(6) The exponent is $\beta = 2$, the spectral range of a single binary distinction: $2 = 1 - (-1)$, the nonzero eigenvalue of $(1 - \tau_i)$ for the self-adjoint involution τ_i at each site (P1's minimal property distinction is binary; this is the one-bit Clifford $\text{Cl}(1,0)$ range under an equivalent name; Comp 100, 102). The numerically equal $\text{KO}_{\text{total}} \bmod 8 = 2$ (Bott periodicity, §1.5) shares the value but enters as a gauge- and spacetime-sector fact, never as an input here; the "KO-tempered" label is retained only as a name. This $\beta = 2$ is also the calibration point of the Bernoulli-MGF route: $\mathbb{E}_\mu[\exp(-\beta\bar{X})] \to e^{-\beta/2}$ hits e^{-1} for $\beta = 2$ and only for $\beta = 2$, so e^{-1} is the signature of the binary gap, and the structural content of this step is precisely the derivation of $\beta = 2$ from P1's binary alphabet.

(7) Combining (1)–(6):

$$\lambda_{\mathrm{SM}}(M_*) \;=\; b \cdot Z_H(\beta_{\mathrm{KO}}) \;\xrightarrow{D\to\infty}\; \frac{\mathrm{e}^{-1}}{4} \;\approx\; 0.0920, \tag{121}$$

structurally. This matches the observed $\lambda_{\mathrm{SM}}(M_*) \approx 0.0927$ [51] at the 0.8% level (within the Buttazzo RGE uncertainty).

Status of the proof sketch. Steps (1), (2), (3), (4), (6) are established in the existing PST framework (§7.21, §1.5, A1 of §7.7, Comp 88). Step (5) is established by Comp 89 (the substrate Higgs Hamiltonian derived from additivity + matched scaling + uniformity, with no free parameters). Step (7) follows from steps (1)–(6). The seven steps are formalised in the partition-function-level Chamseddine-Connes correspondence framework of §7.23, where the *substrate-side* content is closed at proof-detail level: Comp 98 closes items (a) and (b) of the residual gap via the mode-shell block-spin RG fixed-point theorem, and Comp 100 closes item (c) – the spectral-action cutoff identification on the substrate – from P1–P3 alone via tensor-product factorisation forced by P1's Bernoulli product measure. The substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to \mathrm{e}^{-1}$ is therefore a structural result of PST from P1–P3 in the asymptotic $D \to \infty$ limit. The identification of this substrate factor with the SM-side ratio $\lambda_{\mathrm{SM}}(M_*)/b$ is bridge premise (B); as §7.23 sets out, (B) is closed within PST in the substrate limit by a four-step argument (wave-function renormalisation forced into the substrate-factor form, embedding-norm identification, and the unique-soft-mode theorem F11). $Z^2 = \mathrm{e}^{-1}$ is thus a structural theorem in the substrate limit, its one residual being an all-orders equivariant write-up of the decoupling lemma.

The structural-scope theorem-style framing of $Z^2 \approx \mathrm{e}^{-1}$ that emerges from Comps 84–92 reads:

Theorem sketch ($Z^2 \approx \mathrm{e}^{-1}$ bridge). Under the Wilsonian-matching premise that the SM effective Higgs quartic at M_* equals the PST bare LG quartic multiplied by the substrate's KO-tempered Bernoulli partition function – a premise closed within PST in the substrate limit in §7.23 – the identification $\lambda_{\mathrm{SM}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}})$ holds, delivering $Z^2 = \mathrm{e}^{-1}$ structurally in the asymptotic $D \to \infty$ limit.

The numerical prediction $\lambda_{\mathrm{SM}}(M_*) = \mathrm{e}^{-1}/4$ agrees with observation at 0.8%, consistent with the Buttazzo-level SM RGE truncation uncertainty.

7.23. Partition-function-level Chamseddine-Connes correspondence framework

The seven-step proof sketch of §7.22 (Comp 92) identifies the structural ingredients for the Z^2 closure but presents them as a chain of informal steps. This subsection formalises the closure as a unified *partition-function-level Chamseddine- Connes correspondence* framework, in which each step of the proof sketch becomes a definition, proposition, or theorem statement, and the matching is isolated as a single explicit *bridge premise* (B), which is then closed within PST in

the substrate limit by a four-step argument (wave-function renormalisation, embedding-norm identification, and the unique-soft-mode theorem F11).

The framework is parallel to the standard continuum Chamseddine-Connes spectral-action correspondence [50] but operates at the *partition-function* level rather than the *heat-kernel* level, the latter being structurally obstructed for the discrete substrate triple (§7.21, Comps 85, 86). The supporting research-direction work on (B) is documented in: Comp 94 (fluctuation-matching T_{KO} sketch, acknowledged dimensionally suspect, superseded by Comp 95); Comp 95 (reduction of (B) to a single sharper structural claim about the substrate Wilsonian RG flow generator); Comp 96 (Polchinski-style configuration-Hamiltonian Boltzmann averaging on the discrete substrate); Comp 99 (KO-thermal framing held as exploratory and superseded by Comp 100); Comp 102 (structural-independence disambiguation of the two "2"s, KO mod 8 vs one-bit Clifford); and Comp 105 (two-layer one-loop diagram verifying non-double-counting of the M_* zero modes across substrate vs emergent layers).

Definitions.

Definition F1 (PST partition-function spectral triple). A PST partition-function spectral triple is a tuple $(\mathcal{A}, \mathcal{H}, D_{\mathrm{sub}}, \mu, \beta)$ *where* $(\mathcal{A}, \mathcal{H}, D_{\mathrm{sub}})$ *is the substrate spectral triple of §1.5 with* $\mathcal{A}$ *the Boolean* $*$*-algebra on* D*,* $\mathcal{H} = \ell^2(\mathcal{P}(D), \mu)$*,* D_{sub} *the canonical Boolean Dirac operator;* $\mu = \bigotimes_{a \in D} \mathrm{Bern}(\frac{1}{2})$ *is the Bernoulli substrate measure; and* $\beta > 0$ *is the temperature parameter.*

Definition F2 (Partition function of a substrate observable). For an observable $O \colon \mathcal{P}(D) \to \mathbb{R}$*, the* β*-partition function of* O *is*

$$Z(O, \beta) \;:=\; \mathbb{E}_{\mu}\big[\exp(-\beta\, O(C))\big]. \tag{122}$$

Definition F3 (Matched-scaling exponent). For the substrate spectral triple, the matched-scaling exponent is

$$\beta \;:=\; 2, \tag{123}$$

equal to the spectral range of a single binary distinction, $2 = 1 - (-1)$*: each exchange* τ_i *is a self-adjoint involution* ($\tau_i^2 = 1$) *with eigenvalues* $\{+1, -1\}$*, so* $(1 - \tau_i)$ *has nonzero eigenvalue* 2*, forced with no freedom by P1's binary alphabet (equivalently the one-bit Clifford* $\mathrm{Cl}(1, 0)$ *range; Comp 100, 102). The numerically equal* $\mathrm{KO}_{\mathrm{total}} \bmod 8 = 2$ *(with* $\mathrm{KO}_{\mathrm{total}} = 4 + 6 = 10$*, §1.5) shares the value but is a gauge- and spacetime-sector fact, never an input here; the name "KO-temperature"* β_{KO} *is retained below as a label only. This* $\beta = 2$ *is also the calibration point of the Bernoulli-MGF route (the value, and only the value, for which* $\mathbb{E}_{\mu}[\exp(-\beta \bar{X})] \to e^{-1}$*): the* e^{-1} *is the signature of the binary gap.*

Propositions and theorems of the framework.

Proposition F4 (PST EFT below M_*). The PST effective field theory below the matched scale $M_* = 4\pi m_h\sqrt{2/3}$ equals the Standard Model exactly: no PST modes propagate below M_*, and the SM Lagrangian (gauge + Higgs + matter) is the leading Seeley-DeWitt content of the projected spectral action.

Established in §7.21, equation (106).

Proposition F5 (Wilsonian matching at M_*). The IR SM couplings at M_* equal the UV PST couplings transformed by the standard Wilsonian matching condition, expressed as the substrate-side path-integral over modes with momenta in $[M_*, \Lambda]$ where $\Lambda = \sqrt{D}$ is the matched-scaling substrate cutoff.

Established in standard EFT machinery; the substrate-side specification is given in F8 below.

Proposition F6 (PST UV Higgs quartic). The PST UV theory at M_* has bare Higgs quartic coupling $\lambda_{\mathrm{PST}}^{\mathrm{bare}} = b = 1/4$, fixed by the Landau-Ginzburg modal potential coefficient.

Established in §1.5; cf. equation (121).

Proposition F7 (Substrate UV-IR scale separation). At matched scaling $\Lambda = \sqrt{D}$, the substrate UV modes lie in the interval $[M_*, \Lambda]$. The path-integral over these modes contributes the substrate vacuum-fluctuation factor to the SM EFT below M_*.

Established by matched-scaling A1 of §7.7.

Theorem F8 (Substrate Higgs Hamiltonian, Comp 89). The substrate Higgs Hamiltonian $H_{\mathrm{Higgs}}\colon \mathcal{P}(D) \to \mathbb{R}$ is uniquely determined by the three structural constraints (i) additivity (Bernoulli factorisation), (ii) matched scaling (per-site energy $1/D$), and (iii) uniformity (P1 distinguishability symmetry); explicitly,

$$H_{\mathrm{Higgs}}(C) \;=\; \frac{1}{D}\sum_{a\in D} B_a(C) \;=\; \bar{X}(C).$$

No free parameters.

Proof: Comp 89; cf. equation (119).

Theorem F9 (KO-tempered Bernoulli partition function, Comp 88). The partition function of H_{Higgs} at the matched-scaling KO-temperature β_{KO} satisfies

$$Z(H_{\mathrm{Higgs}}, \beta_{\mathrm{KO}}) \;=\; \mathbb{E}_\mu\big[\exp\big(-(\mathrm{KO}_{\mathrm{total}} \bmod 8)\,\bar{X}\big)\big] \;\xrightarrow{D\to\infty}\; \mathrm{e}^{-1}.$$

Proof: standard Cramér-Bernoulli asymptotic; $\mathbb{E}_\mu[\exp(c\bar{X})] \to \exp(c\,\mu_{\mathrm{site}}) = \exp(c/2)$ delivers

e^{-1} at $c = -2 = -(\mathrm{KO}_{\mathrm{total}} \bmod 8)$. Comp 88; cf. equations (117), (118).

The bridge premise and its closure.

Bridge Premise (B). Under the partition-function- level Chamseddine-Connes correspondence, the SM Higgs effective coupling at the matched scale M_* equals the PST bare Higgs quartic $\lambda_{\mathrm{PST}}^{\mathrm{bare}}$ multiplied by the substrate's KO-tempered Bernoulli partition function of H_{Higgs}:

$$\lambda_{\mathrm{SM}}(M_*) \;=\; \lambda_{\mathrm{PST}}^{\mathrm{bare}} \cdot Z(H_{\mathrm{Higgs}}, \beta_{\mathrm{KO}}). \tag{124}$$

Closure of (B) (in the substrate limit). Earlier work left (B) as a structural hypothesis, the partition-function-level analogue of the standard heat-kernel CC correspondence (which the discrete substrate spectrum obstructs; §7.21). It is now derived within PST, in the substrate $D \to \infty$ limit, by a four-step argument; each step is either forced by P1–P3 or a theorem, and the one competing field-theoretic reading is excluded by the finiteness of λ_{SM}. The substrate side $Z(H_{\mathrm{Higgs}}, \beta_{\mathrm{KO}}) \to e^{-1}$ is already closed (F8, F9; no free parameters), and the numerical agreement $(1/4)\, e^{-1} \approx 0.0920$ matches observed $\lambda_{\mathrm{SM}}(M_*) \approx 0.0927$ at the 0.8% Buttazzo-RGE level (Comp 91). The four steps close the *matching* itself:

1. *The bridge factor is a wave-function (field-strength) renormalisation, not a coupling-matching Jacobian.* A field rescaling $\phi \to Z_\phi^{1/2}\phi$ acts *multiplicatively* on the quartic, $\lambda \to \lambda\, Z_\phi^2$, which is the form (B) requires (Comps 101, 110). The measure-Jacobian route is excluded: it is additive, and produces a Gaussian weight $\exp(-\alpha(\bar{X} - 1/2)^2)$, not the substrate factor (Comp 103).

2. *P1's product structure forces the field-strength kernel into the substrate-factor form.* The matched-scaling cutoff factorises over bits (§7.23 above), so the per-configuration field-strength dressing $K(C) = \prod_a g(C_a)$ is a product over the independent bits. A product kernel over Boolean bits is necessarily exp of a quantity *linear* in $\bar{X}$, i.e. $K \propto \exp(-\beta\bar{X})$ – the same form the substrate side evaluates to e^{-1}. The Gaussian of step 1 is thereby *excluded*: it correlates the bits and breaks the product structure (Comp 111). Hence the wave-function route is unobstructed, in contrast to the Jacobian route.

3. *The field-strength is the embedding norm of the symmetric Higgs vacuum, equal to the substrate factor; the standard-EFT alternative is excluded by the data.* For the Higgs as an emergent projected field $\phi = \Pi(\psi)$, the field-strength is its norm inside the substrate Hilbert space under the modal (tempered) inner product, $Z_\phi^2 = \mathbb{E}_\mu[\exp(-\beta_{\mathrm{KO}}\bar{X})] \to e^{-1}$. The alternative – the standard-EFT two-point residue of $\bar{X}$ – is $\mathrm{Var}_\mu(\bar{X}) = 1/(4D) \to 0$, which would give $\lambda_{\mathrm{SM}} \to 0$ and is excluded by the finiteness of the observed quartic (Comp 114). The embedding-norm reading holds precisely when the substrate→Higgs reduction is *total* (every substrate mode folded into the single Higgs sector), which step 4

establishes.

4. *The reduction is total: a unique-soft-mode theorem.* Past the Landau-Ginzburg threshold (P3) the order parameter $\bar{X} = (1/D)\sum_a C_a$ is the unique soft mode, with a spectral gap (the binary gap $= M_*$) to every other substrate mode. This is Theorem F11 below; combined with step 3 it delivers $Z_\phi^2 \to \mathrm{e}^{-1}$ and hence (B).

The residual is finite-D: an $O(1/D)$ constraint correction that vanishes in the substrate limit (the same limit in which e^{-1} itself is defined). The matched scaling $\Lambda^2 = D$ is the one definitional identification shared with the substrate side.

Theorem F11 (unique soft mode; total reduction). Let the substrate action be $\Delta + V(\bar{X})$, where $\Delta = \sum_a (1 - \tau_i)$ is the ultralocal Boolean Laplacian (P1: independent bits) and V is the Landau-Ginzburg potential, a function of the linear collective coordinate $\bar{X} = (1/D)\sum_a C_a$ alone (Theorem F8: $H_{\mathrm{Higgs}} = \bar{X}$). Then, past the LG threshold (P3), $\bar{X}$ is the unique soft mode: every non-collective substrate mode (the S_D standard representation in the $|S| = 1$ shell, and all $|S| \geq 2$ Walsh shells) retains the free binary-gap mass ≥ 2 up to $O(1/D)$, while V shifts only the S_D-singlet direction $\bar{X}$. Hence below M_* the single surviving light field is the Higgs, the substrate→Higgs reduction is total, and the field-strength of step 3 is the full substrate embedding norm.

Proof sketch: because $\bar{X}$ is *linear* in the bits, every derivative tensor of $V(\bar{X})$ is the totally symmetric all-ones tensor $V^{(n)}(\bar{X})/D^n$, which has zero matrix element on the non-collective subspace at every order n. The non-collective modes therefore never appear in V: they are free fields, decoupled from the interacting collective sector, with mass set by Δ alone (the binary gap; §7.23). No order of V and no loop built from V-vertices can move them. At quadratic order the LG Hessian is rank one (it acts only on the symmetric direction $= \bar{X}$), softening $\bar{X}$ to criticality at threshold while the $D - 1$ standard-rep modes stay at the binary gap. Beyond quadratic order the only coupling between bits is the $O(1/D)$ sum-constraint correlation, which vanishes as $D \to \infty$; the per-pair coupling the non-collective modes feel is $O(1/D)$, while the singlet susceptibility accumulates the full $O(1)$ effect. Thus the spectral gap is protected non-perturbatively in the substrate limit. (Comps 115, 116.) □

Main theorem of the framework.

Theorem F10 ($Z^2 \to \mathrm{e}^{-1}$ structural closure in the substrate limit). In the substrate $D \to \infty$ limit, with Bridge Premise (B) established by the four-step closure above (steps 1–3 forced/excluded, step 4 = Theorem F11), the PST framework delivers the asymptotic identification

$$\lambda_{\mathrm{SM}}(M_*) \;=\; b \cdot Z(H_{\mathrm{Higgs}}, \beta_{\mathrm{KO}}) \;\xrightarrow{D\to\infty}\; \frac{\mathrm{e}^{-1}}{4} \;=\; \frac{1}{4\mathrm{e}} \;\approx\; 0.0920, \tag{125}$$

equivalent to

$$Z^2 \;=\; \frac{\lambda_{\mathrm{SM}}(M_*)}{b(M_*)} \;\xrightarrow{D\to\infty}\; \mathrm{e}^{-1},$$

with the substrate-side fact $Z(H_{\mathrm{Higgs}}, \beta_{\mathrm{KO}}) = \mathrm{e}^{-1}$ *structurally derived from P1 (*$\mu_{\mathrm{site}} = 1/2$*), the binary-distinction gap* $\beta_{\mathrm{KO}} = 1 - (-1) = 2$*, and the matched-scaling limit, with no free parameters.*

Proof: combining propositions F4 (PST EFT = SM below M_*), F5 (Wilsonian matching), F6 ($\lambda_{\mathrm{PST}}^{\mathrm{bare}} = b$), and F7 (UV scale separation) with theorems F8 ($H_{\mathrm{Higgs}} = \bar{X}$) and F9 (KO-tempered partition function $\to \mathrm{e}^{-1}$), the four-step closure of (B) (steps 1–3 and Theorem F11) yields equation (125). □

Remark on what the framework achieves. Theorem F10, together with the four-step closure of (B), derives $Z^2 \to \mathrm{e}^{-1}$ from P1–P3 in the substrate limit, in the language of a partition-function-level analogue of the Chamseddine-Connes correspondence [49, 50] adapted to the discrete substrate triple. Every structural input is derived from PST postulates. The framework therefore:

- closes the matching premise (B) within PST, not by inheriting an SM-side CC postulate, via the wave-function-renormalisation route (steps 1–3) and the unique-soft-mode theorem (F11);
- separates the substrate-side closure (F8, F9, exact) from the matching (B), and closes both;
- provides a unified mathematical statement of the Z^2 identification in the language of partition functions on substrate observables;
- leaves a single rigour task – the all-orders equivariant statement of the decoupling lemma (F11) – and the standard $O(1/D)$ substrate-limit caveat;
- documents what is and is not established at each step.

Substrate-side closure and matching closure. The tensor-product factorisation (Comp 100) and the mode-shell block-spin fixed-point theorem (Comp 98) close the *substrate-side* content: P1's Bernoulli product measure induces a tensor-product decomposition $L^2(\mathcal{P}(D), \mu) = \bigotimes_{i=1}^{D} L^2(\{0,1\}, \mu_i)$ on the substrate Hilbert space; the spectral-action trace's compatibility with this independence structure forces the cutoff function to $f(x) = \exp(-x)$ (Comp 100); and at the matched scaling $\Lambda^2 = D$, binomial product convergence delivers $(1/2^D)\,\mathrm{Tr}\, e^{-\Delta/D} \to e^{-1}$ asymptotically. The *matching* form $\lambda_{\mathrm{SM}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}})$ – bridge premise (B), eq. (124) – is then closed by the four-step argument above: it is a wave-function renormalisation (multiplicative, not the additive Wilsonian quartic matching, Comp 103), whose field-strength is forced by P1's product structure into the substrate-factor form (Comp 111) and equals the embedding

norm of the symmetric Higgs vacuum (Comp 114), with the total reduction underwriting that identification supplied by the unique-soft-mode theorem F11 (Comps 115, 116). Both the substrate side and the matching are thus closed in the substrate limit; the remaining work is the all-orders equivariant write-up of F11's decoupling lemma.

7.24. Summary

This section derives Einstein's field equations as consequences of the projection of equation (40) from the substrate, with the kernel normalisation $\mathcal{N}$ determined in §7.9 from the Bernoulli measure. The stress-energy tensor turns out to be the same precausal tension that produced the spacetime geometry, now expressed in geometric units. Gravity and matter are not two separate things connected by the field equations; they are two faces of one underlying structure. The section also shows that the vacuum (the ground state of the universe after sublimation) has the shape of a ring rather than a single point in an internal sense. §7.19 derives the angular momentum tensor from the Goldstone kinetic term, identifies localized bodies as vortex configurations with topological winding number n, and establishes $L = n\hbar$ from first principles, including the PST elementary mass scale $m_0 = 8\pi^3 r_0^2 d_0/(3c^2)$. The interacting vacuum is stable at one loop and reproduces the Standard Model as its Wilsonian EFT below M_*, with the Higgs mass nominally relating to the modal scale via $m_h = M_*\sqrt{6}/(8\pi)$ at the LG quartic $\lambda_{\rm PST} = \frac{1}{4}$. The numerical value $M_* = 4\pi m_h\sqrt{2/3} \approx 1285$ GeV is a parameter-free one-loop derivation fixed by the single measured input m_h: the earlier off-by-one (Computations 67, 69) is closed by the substrate-vs-emergent zero-mode distinction (Computations 93, 97), with N_B (including the substrate Walsh zero mode) and C_H (the emergent Higgs doublet) counted in structurally distinct layers of the projection chain. The coherence length d_0 and the projection coefficient $\kappa_{\rm proj}$ are Landau-Ginzburg/condensate quantities whose magnitudes are not pinned by the postulates alone.

Claim status of § 7. *Derived theorems (under conditions noted)*: stress-energy = projected tension; Einstein-Hilbert action as leading Seeley-DeWitt term; Newton's G as a consistency relation among (m_h, M_P, v) rather than a free input. *Identification (§10.4)*: $\phi_{\rm Higgs} = \Pi(\psi)$ – the SM Higgs field and the modal order parameter are the same field at different scales, with $m_h = 125$ GeV recovered via the LG quartic. *Parameter-free one-loop derivation of the scalar self-mass (Computations 67, 69, 93, 97)*: $M_* = 4\pi m_h\sqrt{2/3} \approx 1.29$ TeV. The apparent zero-mode off-by-one of the boson–fermion cancellation (Comps 67, 69) is closed by the substrate-vs-emergent distinction (Comps 93, 97): the substrate Walsh zero mode $\chi_\emptyset$ identifies with the vacuum/VEV direction (constant function on M), structurally distinct from the emergent 4-component SM Higgs doublet generated by inner fluctuations on the spectral triple $M \times F$ – so the rescue is a VEV-vs-fluctuation split, with the projected zero mode being $\langle\phi\rangle$ rather than the physical h. Including the zero mode in N_B does not double-count $C_H = 6\lambda = 3/2$, and the scalar self-mass relation $m_h^2 = (3/2)\, M_*^2/(16\pi^2)$ recovers as a

parameter-free one-loop derivation *of the scalar sector*. The extension of the boson–fermion cancellation to the full SM (top/W/Z sectors) requires the configuration-to-particle map, cross-sector coupling universality at M_*, and the inner-fluctuation sign assignment of §7.21, and remains a structural conjecture not demonstrated here — in tension, moreover, with the large negative Veltman residual of equation (104). $Z^2 \approx e^{-1}$ *status (closed in the substrate limit)* (§7.21, §7.22, §7.23, Computations 71, 84–100): the substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ is derived structurally from P1–P3, and its identification with the SM-side ratio $\lambda_{\mathrm{SM}}(M_*)/b$ – bridge premise (B) – is closed within PST in the substrate limit, the matching being a wave-function renormalisation forced into the substrate-factor form with the total reduction supplied by the unique-soft-mode theorem (§7.23). The 0.8% empirical agreement holds at full Buttazzo-level two-loop SM RGE precision (and 5–7% at one-loop precision, Comp 91). The Chamseddine-Connes inner-fluctuation route is structurally obstructed for the discrete substrate triple (Comps 85, 86: heat-kernel expansion collapse); the Bernoulli moment-generating function at scale $c = -(\mathrm{KO}_{\mathrm{total}} \bmod 8) = -2$ delivers e^{-1} asymptotically (Comp 88) with all structural ingredients derived from existing PST axioms. The substrate Higgs Hamiltonian $H_{\mathrm{Higgs}} = \bar{X}$ follows from substrate primitives (Comp 89, no free parameters); the seven-step Wilsonian-matching proof sketch delivers $\lambda_{\mathrm{SM}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}}) \to \mathrm{e}^{-1}/4 \approx 0.0920$ structurally (Comp 92, equation (121)), matching observed 0.0927 at the same 0.8% level. The seven-step proof sketch is formalised as the partition-function-level Chamseddine-Connes correspondence framework of §7.23, isolating the matching as a single explicit Bridge Premise (B) (equation (124)) – the partition-function-level analogue of the standard heat-kernel CC correspondence for the discrete substrate triple. Computation 94 attempts the derivation of (B) via substrate-measure invariance with the fluctuation-matching prescription $\langle(\bar{X} - 1/2)^2\rangle_\mu = \langle h^2\rangle_{\mathrm{thermal},T}$, giving the matched-scaling temperature $T_{\mathrm{KO}} = \mu^2(M_*)/(4D)$ and, combined with $\beta_{\mathrm{KO}} = 2$ from the binary-distinction gap $1 - (-1)$ (P1), the specific testable prediction $\mu^2(M_*) = 2D \cdot M_*^2$ for the Higgs mass-squared at the matched scale in terms of substrate size D. This was an early route toward (B); the closure ultimately follows the wave-function-renormalisation argument below (Comps 110–116). Computation 95 provides a parallel reduction via substrate-Wilsonian RG: Bridge Premise (B) is equivalent to the structural claim (***) that the substrate Wilsonian RG flow generator equals the Boltzmann partition function $\exp(-\beta_{\mathrm{KO}}\, H_{\mathrm{Higgs}}(C))$ on substrate configurations – the partition-function-level analogue of the heat-kernel RG flow generator $\exp(-tD^2)$ for the continuum spectral triple. Computation 96 attempts a Polchinski-style derivation of (***) on the discrete substrate, establishing the partition-function form structurally via configuration-Hamiltonian Boltzmann averaging $\mathbb{E}_\mu[\exp(-\beta_{\mathrm{KO}}\bar{X})] \to \mathrm{e}^{-1}$ asymptotically, and identifies three rigorous-proof items as the residual gap: (a) a formal exact RG flow equation on the discrete substrate triple, (b) a uniqueness theorem for the Boltzmann form at matched scaling, and (c) the cutoff-function identification with the asymptotic $D \to \infty$ limit. Computation 98 closes (a) and (b) at proof-detail level via an explicit discrete Wilson block-spin construction. The key observation collapses both items together: the substrate

Higgs Hamiltonian $H_{\text{Higgs}} = \bar{X}$ (Comp 89) is supported on the lowest two Walsh-mode shells $V_0 \oplus V_1$ (the zero mode plus the singleton modes), with explicit Walsh-coefficient expansion $\bar{X} = \frac{1}{2}\chi_\emptyset - \frac{1}{2D}\sum_{|S|=1}\chi_S$. Because $\bar{X}$ has no $|S| \geq 2$ content, the block-spin RG step from level $m+1$ to m factorises trivially: $\bar{X}(\psi_{\leq m} + \psi_{m+1}) = \bar{X}(\psi_{\leq m})$ for every $m \geq 1$, so the Boltzmann action $\beta_{\text{KO}}\bar{X}$ is preserved exactly with coefficient unchanged. This is Theorem 1 of Comp 98: the Boltzmann form is a *trivial fixed point* of the substrate block-spin RG via mode-shell localisation, and the form is unique among block-spin fixed points supported on $V_{\leq 1}$. Combined with Comp 89's boundary condition at $\Lambda = \sqrt{D}$ (which forces $H_{\text{Higgs}} = \bar{X}$ structurally from additivity, matched scaling, and uniformity), items (a) and (b) are closed at proof-detail level. Computation 100 closes the residual item (c) – the substrate-side matched-scaling cutoff identification – from P1–P3 via a tensor-product structural argument. (The bridge to the SM-side coupling ratio, i.e. the matching identity $\lambda_{\text{SM}}(M_*) = b \cdot Z_H(\beta_{\text{KO}})$ itself – bridge premise (B) – is a separate, still-open step, addressed at the end of this section.) P1's Bernoulli independence gives the tensor product $L^2(\mathcal{P}(D), \mu) = \bigotimes_{i=1}^{D} L^2(\{0,1\}, \mu_i)$; the Boolean Laplacian decomposes as $\Delta = \sum_i (1 - \tau_i)$, where each exchange τ_i on a single binary slot is a self-adjoint involution ($\tau_i^2 = 1$), hence has spectrum $\{+1, -1\}$, so $(1 - \tau_i)$ has eigenvalues $\{0, 2\}$. The load-bearing factor 2 is therefore the most elementary fact about P1's alphabet: it is the spectral range of a single binary distinction, $2 = 1 - (-1)$, the difference between "present" and "absent". P1 posits property differentiation, whose minimal form is binary, so this gap is *forced*, with no freedom: a single self-adjoint involution admits no other range. (Algebraically this is the one-bit Clifford $\text{Cl}(1,0)$ eigenvalue range, an equivalent name for the same binary gap; it is independent of the KO-dimension value $4 + 6 \equiv 2 \pmod 8$, which shares the number 2 but enters as a gauge- and spacetime-sector fact, never as an input to this factor. Only the binary gap is used here.) The spectral-action trace $\text{Tr}\, f(\Delta/\Lambda^2)$ must respect the substrate's independence structure: a path integral $\int e^{-S_{\text{eff}}} d\mu$ over independent bits factorises iff the effective action is additive over bits, which at the spectral-action level $S_{\text{eff}} = \text{Tr}\, f(\Delta/\Lambda^2)$ translates to the requirement that the trace factor as a product of one-bit traces. This is not a separate framework postulate – the requirement is forced by P1's product measure μ: a non-factoring spectral action would couple modes that are by construction independent under μ, contradicting P1. Factorisation $\text{Tr}\, f(\sum_i A_i) = \prod_i \text{tr}_i\, f(A_i)$ for commuting one-bit operators A_i requires $f(x+y) = f(x)f(y)$, whose unique smooth solution with $f(0) = 1$ and decay is $f(x) = \exp(-x)$. The cutoff function is therefore *determined* by P1's product structure, not chosen as in standard CC. At matched scaling $\Lambda^2 = D$ (substrate dimension, the natural matched scale for Δ's spectrum):

$$\frac{1}{2^D}\text{Tr}\, e^{-\Delta/D} = \prod_{i=1}^{D} \frac{1 + e^{-2/D}}{2} = \left(\frac{1 + e^{-2/D}}{2}\right)^D \xrightarrow{D\to\infty} e^{-1} \tag{126}$$

asymptotically by binomial product convergence. This delivers the substrate-side factor

$Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ using only Bernoulli product structure (P1), the binary-distinction gap $2 = 1 - (-1)$ (equivalently the one-bit Clifford range, forced by P1's binary alphabet), and matched scaling at substrate dimension D – no separate CC framework postulate. Bridge premise (B) – the identification of this substrate factor with the SM-side ratio $\lambda_{\mathrm{SM}}(M_*)/b$ – is closed within PST in the substrate limit by the four-step argument above (wave-function renormalisation forced into the substrate-factor form, embedding-norm identification, and the unique-soft-mode theorem F11); $\lambda_{\mathrm{SM}}(M_*) = b \cdot e^{-1} = 0.0920$, matching observed 0.0927 at full-Buttazzo-level two-loop SM RGE precision (5–7% at one-loop, Comp 91). The "v follows from m_h" reduction inherits the same status. *A6 equicoercivity closed at proof-detail level*: the Mosco-convergence theorem of §7.7 is stated conditionally on six assumptions A1–A6; A6 is reduced in §7.7 to the uniform-rate hypothesis (R), and the three closure subsections (a)–(c) of §7.7 (explicit $\Pi_n^\dagger$ construction, thin-support d_0-corrections, Riemannian extension on $M = \mathbb{R}\times S^3$) deliver (R) at proof-detail level. Computations 70 and 83 verify the rate empirically across single and seven-family test sets. The remaining work is explicit numerical-constant computation in the Sobolev embeddings (standard finite-element analysis, Brenner-Scott [101]); no fresh structural content survives.

8. Gauge Structure, Quantization, and the Standard Model

8.1. From U(1) to the full Standard Model gauge group

The projected field theory of Chapter 7 establishes a Lorentzian scalar field theory carrying a topologically quantised U(1) charge, with a massive radial mode (the Higgs) and a massless Goldstone mode. That treatment was explicit about two structural gaps: the step from global to local gauge invariance was deferred, and the non-Abelian factors $\mathrm{SU}(2)_L \times \mathrm{SU}(3)_c$ of the Standard Model were expected but not derived. The present section closes both gaps and establishes a third result: gauge anomaly cancellation is not a numerical coincidence of the Standard Model but a structural theorem whose proof rests on PST's $N_c = 3$, the dimension of the colour factor $M_3(\mathbb{C})$ of the internal algebra.

The internal algebra itself, $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, is not posited in this section but *derived* from P1–P3 in §8.8: the substrate's parity-selected real structure delivers the chirality grading $\gamma_K = i\omega_{\mathrm{JW}}$ exactly in the Jordan-Wigner representation (Computation 19), so the chirality projector $f = (I + i\omega)/2$ is the substrate's natural primitive idempotent, and Furey 2014 [120] delivers A_F as the commutant of the matter representation on the minimal left ideal $\mathrm{Cl}(0,6)\cdot f$. The Standard-Model gauge group then emerges as the unimodular unitary group of A_F, without the algebra identity being inherited from the Chamseddine-Connes-Marcolli classification. The full derivation is carried out in Comp 6 (gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ as the unimodular unitary group of A_F; inner-fluctuation count delivering 12 gauge bosons plus the Higgs scalar). Together, these results constitute the gauge-theoretic skeleton of a complete PST derivation of

the Standard Model.

8.2. Path-integral quantization of the projected theory

The Euclidean modal functional $\mathcal{F}[\psi, C]$, equipped with the Bernoulli product measure μ derived in Chapter 4, defines a substrate partition function:

$$Z_{\text{sub}} \;=\; \int_{\mathcal{C}} \exp\bigl(-\mathcal{F}[\psi(C), C]\bigr)\, d\mu(C). \tag{127}$$

The functor $\Phi : \mathbf{S} \to \mathbf{G}$ maps each configuration to an instantiated spacetime geometry; the F–S dichotomy of Chapter 7 establishes that this is the precausal counterpart of Wick rotation. The instantiated partition function is therefore:

$$Z_{\text{QFT}} \;=\; \int \mathcal{D}\psi\, \mathcal{D}g_{\mu\nu}\, \exp\bigl(iS[\psi, g]\bigr), \tag{128}$$

with $S = S_{\text{EH}} + S_{\text{matter}}$ the Einstein-Hilbert plus matter action of Chapter 7. This is the path integral of quantum gravity coupled to the modal field. The substrate grain d_0 derived in Chapter 10 provides a physical ultraviolet cutoff $\Lambda_{\text{UV}} = 1/d_0$: not a calculational regularisation but a genuine breakdown scale of the continuum description. Renormalisability is the statement that low-energy observables are insensitive to d_0, which holds for separations $\gg 7$ nm exactly as the Casimir analysis requires.

Expanding around the vacuum $\psi = r_0 e^{i\theta_0}$, the quadratic fluctuation action S_2 of equation (102) generates Feynman rules for a massive scalar (mass $m_h^2 = 4\varepsilon$, equation (101)) and a massless pseudoscalar (the Goldstone mode [43]), together with graviton propagators from the metric fluctuations $h_{\mu\nu}$. The non-Abelian gauge bosons enter through the extended order-parameter structure of the paragraphs below.

8.3. From global to local gauge symmetry: the photon as a connection

The global U(1) symmetry $\psi \mapsto e^{i\alpha}\psi$ is a symmetry of the action S_2 with α a single number. But the projection Π maps the continuum of substrate configurations to the full manifold (M, g): different loci of M are images of different configurations C, each of which independently selects a vacuum phase $\theta(x) \in \mathcal{V} \cong S^1$. The symmetry parameter is therefore an arbitrary smooth function $\alpha(x)$, and the symmetry is local.

When α depends on x, the kinetic term $r_0^2\, \partial_\mu\theta\, \partial^\mu\theta$ is not invariant: under $\theta(x) \mapsto \theta(x) + \alpha(x)$ the derivative picks up $\partial_\mu\alpha(x)$. The geometric resolution is that $\theta(x)$ is a section of a U(1) principal bundle over M, and the correct derivative is the gauge-covariant one $D_\mu\theta = \partial_\mu\theta - A_\mu$, where $A_\mu(x)$ is the connection (the photon field). Under $A_\mu \mapsto A_\mu + \partial_\mu\alpha(x)$, the covariant derivative is invariant. The field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature of the connection. Integrating out short-wavelength fluctuations of θ in Z_{QFT} generates the Maxwell kinetic term

at quadratic order in A_μ:

$$S_{\text{Maxwell}} \;=\; -\frac{1}{4e^2}\int_M F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\;d^4x, \tag{129}$$

with gauge coupling $e^{-2} \propto r_0^2$. The photon is exactly massless because A_μ transforms inhomogeneously and cannot acquire a gauge-invariant mass term; minimal coupling of charged fields to A_μ via D_μ follows from gauge covariance of the projected kinetic terms. The gauging is not an additional postulate: it is the consequence of Π assigning an independent vacuum phase to each spacetime point.

8.4. $\mathrm{SU}(2)_L$ from the threshold topology: the electroweak doublet

The single complex field ψ is the leading-order description of the modal order parameter. At the electroweak scale the full threshold structure must be represented. The threshold τ is a hypersurface in $\mathcal{C}$; near any crossing point the configuration explores the level set $T(C) = \tau$. The minimal order parameter consistent with both $\mathrm{U}(1)_Y$ and $\mathrm{SU}(2)_L$ is a complex doublet:

$$H \;=\; \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \;\in\; \mathbb{C}^2, \qquad |H|^2 \;=\; |H^+|^2 + |H^0|^2. \tag{130}$$

The scalar ψ of Chapter 7 is the neutral component H^0 in the broken phase; H is the Standard Model Higgs doublet [75, 76].

For $H \in \mathbb{C}^2 \cong \mathbb{R}^4$, the vacuum manifold after crossing is:

$$\mathcal{V}_{\text{EW}} \;=\; \left\{\, H \in \mathbb{C}^2 \;:\; |H|^2 = v^2/2 \,\right\} \;\cong\; S^3, \tag{131}$$

where $v = r_0\sqrt{2} = 246$ GeV is the electroweak vev [93]. The three-sphere S^3 is diffeomorphic to the group manifold of SU(2):

$$S^3 \;\cong\; \mathrm{SU}(2), \tag{132}$$

and its isometry group is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. The action $H \mapsto UH$ with $U \in \mathrm{SU}(2)$ preserves $|H|^2$ and is the electroweak $\mathrm{SU}(2)_L$ symmetry. Three of the four real degrees of freedom in H are Goldstone bosons of the breaking $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\text{em}}$; they are absorbed by the $W^\pm$ and Z bosons, giving them mass [77, 78, 79]. The remaining degree of freedom is the physical Higgs boson of mass $m_h^2 = 4\varepsilon$, identified in equation (101).

The $\mathrm{SU}(2)_R$ isometry is broken by the directional character of the threshold crossing: $\varepsilon = T(C) - \tau > 0$ is an oriented statement in which tension exceeds the threshold from below, selecting a preferred chiral handedness. This asymmetry is the origin of the L-subscript on $\mathrm{SU}(2)_L$: the weak interaction couples only to left-handed fermions because the threshold crossing is an irreversibly directed, parity-violating process. Parity violation in the weak sector

is therefore not an external assumption but a theorem of the asymmetric modal threshold [83, 84].

Three viewpoints of one Goldstone phase. The single LG phase mode θ on the substrate side appears in three formally different but physically identical guises across the energy hierarchy, all describing the same field at the same scale viewed in different language:

1. *Substrate / Stueckelberg view* (§8.3): θ is the $\mathrm{U}(1)_{\mathrm{em}}$ gauge phase of ψ with covariant derivative $D_\mu\theta = \partial_\mu\theta - A_\mu$. Because $\mathrm{U}(1)_{\mathrm{em}}$ remains unbroken in the IR (the photon stays massless), θ is Stueckelberg-eliminable by gauge fixing: it is not an independent propagating physical scalar but a gauge redundancy of the $\mathrm{U}(1)_{\mathrm{em}}$ description.
2. *Electroweak doublet view* (this section): θ is the $\mathrm{U}(1)_{\mathrm{em}}$ phase of H^0 once $\psi = H^0$ is enriched to the SM Higgs doublet $H = (H^+, H^0)^{\mathrm{T}}$. The three angular Goldstones of H are the eaten dofs of the $\mathrm{SU}(2)_L\times\ \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{em}}$ breaking, absorbed into the longitudinal polarisations of $W^\pm$ and Z; the $\mathrm{U}(1)_{\mathrm{em}}$ phase of H^0 is the unbroken-direction gauge redundancy, the same redundancy as point (1).
3. *Cosmological zero-mode view* (§11.5): A spatially uniform rotation $\psi = r_0 e^{i\omega t}$ would carry Noether charge $J^0 = 4\varepsilon\omega$, the global $\mathrm{U}(1)_{\mathrm{em}}$ charge of the universe. The observed cosmic charge-neutrality pins $\omega = 0$, consistent with the gauge-eliminable nature of θ in views (1) and (2).

These three are not three different fields nor three different mechanisms; they are one Goldstone phase described in three vocabularies appropriate to three observational regimes (IR, electroweak, cosmological). The unifying statement is: θ does not propagate as an independent physical scalar in emergent spacetime. The only propagating scalar dof of the LG–Higgs sector is the radial mode, the physical Higgs h of mass $m_h \simeq 125$ GeV.

Relation to existing approaches to electroweak symmetry breaking. How PST's substrate-Higgs mechanism (this section together with the identification §10.4 and the gauging §8.3) compares against the Connes–Chamseddine–Marcolli spectral action, Technicolor and composite-Higgs models, Coleman–Weinberg radiative SSB, Little Higgs models, and supersymmetric extended Higgs sectors is treated in §12.13.

Chirality of $\mathrm{SU}(2)_L$. The derivation above gives the $\mathrm{SU}(2)_L$ doublet but not its chirality (its action on left-handed components only). Chirality is a separate structural consequence of the directed modal threshold, treated in §8.21. The same threshold-direction parity argument that drives parity violation selects which of the two su(2) factors of $\mathrm{so}(4) = \mathrm{su}(2)_L \oplus \mathrm{su}(2)_R$ on the spatial S^3 is identified with the physical $\mathrm{SU}(2)_L$.

8.5. $\mathrm{SU}(3)_c$ as the colour factor of the internal algebra

Colour $\mathrm{SU}(3)_c$ is the symmetry of the $M_3(\mathbb{C})$ summand of the internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The underlying Furey-2014 foundation – the action of the octonion chain algebra $\overleftarrow{\mathbb{O}}$ on one-generation matter content delivering the SU(3) structure – is verified in Comp 16. The Standard Model gauge group is the unimodular unitary group of A_F (§13.10), with $\mathrm{SU}(3)_c$ the unimodular unitaries of the $M_3(\mathbb{C})$ factor, $\mathrm{SU}(2)_L$ the unit quaternions of $\mathbb{H}$, and $\mathrm{U}(1)_Y$ the surviving abelian factor. Equivalently, the colour group is the automorphism group of the colour vector space $\mathbb{C}^3$ preserving its Hermitian structure:

$$\mathrm{Aut}\big(\mathbb{C}^3,\, \langle\cdot,\cdot\rangle\big) \;=\; \mathrm{U}(3) \;=\; \mathrm{U}(1) \times \mathrm{SU}(3)_c, \tag{133}$$

with the overall U(1) removed by unimodularity.

The octonionic origin is explicit. The complexified octonions $\mathbb{C} \otimes \mathbb{O}$ generate $\mathrm{Cl}(6) \cong M_8(\mathbb{C})$, whose minimal left ideal carries one generation's colour and electric-charge content (§8.18); the exceptional group $G_2 = \mathrm{Aut}(\mathbb{O})$ contains SU(3) as the stabiliser of a unit vector in V_7, and the three complex directions of $\mathrm{Im}(\mathbb{O})$ on which it acts are the three colours. The colour count $N_c = 3$ is therefore the rank of the $M_3(\mathbb{C})$ factor.

Consequences of $N_c = 3$. *(i)* Quark charges are fractional: a colour-neutral baryon of $N_c = 3$ quarks must carry integer electric charge, forcing each quark to carry charge $\pm e/3$ or $\pm 2e/3$. *(ii)* Colour confinement at low energies follows from the non-Abelian structure of any $\mathrm{SU}(N_c)$ gauge theory with $N_c \geq 2$, inherited by PST from the standard QCD argument [81, 82, 80].

8.6. Octonionic SU(3) inside Cl(0, 6): rigorous Casimir decomposition

The qualitative octonionic-origin story of the previous paragraph admits a rigorous, fully checked algebraic version. The Lie group $G_2 = \mathrm{Aut}(\mathbb{O})$ acts on the direction algebra $V_7 \cong \mathbb{R}^7$; fixing a unit direction $e_7 \in V_7$ singles out the stabiliser $\mathrm{Stab}_{G_2}(e_7) = \mathrm{SU}(3)$, the octonionic colour group. The associated complex structure on the six-dimensional complement is $J = L_{e_7}$, octonionic left multiplication by e_7. In the standard Cayley convention used throughout this paper, J pairs the basis directions as (e_1, e_3), (e_2, e_6), (e_4, e_5) (Computation 16 §16), and the $J = +i$ eigenspace is the SU(3) triplet

$$T \;=\; \mathrm{span}_{\mathbb{C}}\{u_1, u_2, u_3\}, \qquad u_1 = \tfrac{1}{\sqrt{2}}(e_1 - ie_3),\; u_2 = \tfrac{1}{\sqrt{2}}(e_2 + ie_6),\; u_3 = \tfrac{1}{\sqrt{2}}(e_4 - ie_5),$$

with its $J = -i$ conjugate $\bar{T} = \mathrm{span}_{\mathbb{C}}\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ giving the anti-triplet. The complex octonion space therefore decomposes under SU(3) as

$$\mathbb{C}^8 \;\cong\; \langle e_0\rangle \oplus \langle e_7\rangle \oplus T \oplus \bar{T} \;=\; \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}. \tag{134}$$

The structural claim about §8.5's qualitative identification is then a sharp Clebsch–Gordan computation, fully verified in Computation 17 (`computation_17.py`). Lifting each Gell-Mann λ_a to an 8×8 matrix Λ_a that acts as λ_a on T, as $-\lambda_a^T$ on $\bar{T}$ (the conjugate representation), and as zero on the two singlets, the SU(3) commutation relations $[\Lambda_a, \Lambda_b] = 2if_{abc}\Lambda_c$ hold to machine precision. The adjoint action of SU(3) on $\overleftarrow{\mathbb{O}} \cong M_8(\mathbb{C})$ is then $\mathbf{8} \otimes \overline{\mathbf{8}}$, and the $(\mathbf{1} + \mathbf{1} + \mathbf{3} + \overline{\mathbf{3}}) \otimes (\mathbf{1} + \mathbf{1} + \overline{\mathbf{3}} + \mathbf{3})$ Clebsch–Gordan expansion gives

$$M_8(\mathbb{C}) \;=\; 6 \cdot \mathbf{1} \oplus 5 \cdot \mathbf{3} \oplus 5 \cdot \overline{\mathbf{3}} \oplus 2 \cdot \mathbf{8} \oplus \mathbf{6} \oplus \overline{\mathbf{6}}. \tag{135}$$

Dimension check: $6 \cdot 1 + 5 \cdot 3 + 5 \cdot 3 + 2 \cdot 8 + 6 + 6 = 64$.

To verify equation (135) numerically, Computation 17 constructs the SU(3) quadratic Casimir $C_2 = \sum_a (\mathrm{ad}\,\Lambda_a)^2/4$ as a 64×64 Hermitian matrix and diagonalises it. The Casimir is an irrep discriminator: it takes the values 0 on $\mathbf{1}$, 4/3 on $\mathbf{3}$ and $\overline{\mathbf{3}}$, 3 on $\mathbf{8}$, and 10/3 on $\mathbf{6}$ and $\overline{\mathbf{6}}$. The numerical Casimir spectrum is

C_2 value	irrep dimension	expected mult.	found mult.
0	1	6	6
4/3	3	30	30
3	8	16	16
10/3	6	12	12

to machine precision, total $64 = 64$. The decomposition of equation (135) is therefore not a conjecture but an exact Clebsch–Gordan identity verified at the irrep-multiplicity level.

Structural and contingent parts of the three-generation question. The verified decomposition of equation (135) settles the *gauge-group* structural derivation: the colour SU(3) is the unique octonionic Lie subgroup of G_2, it acts on PST's substrate Clifford algebra $\mathbb{C} \otimes \mathrm{Cl}(0,6) \cong M_8(\mathbb{C})$ through the natural representation (134), and the irrep content of the substrate algebra matches the Clebsch–Gordan expectation exactly. No separate ansatz puts the colour SU(3) into the internal algebra; it is already there.

This is the structural closure of the gauge-group portion of the algebra-identity question. It does *not* by itself fix $N_{\mathrm{gen}} = 3$ generations: equation (135) gives the colour decomposition of $M_8(\mathbb{C})$ but is silent on the generation count. The generation count is fixed separately, by the V_7-directional structure of P1–P3 and the $\hat{\tau}$-selection at modal sublimation: the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$ deliver three disjoint generation sectors of $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, each carrying one generation's Standard-Model matter content (§8.19, Computations 46–48).

8.7. Consistency of the octonionic interpretation across PST's structural results

The colour-block embedding $\mathrm{SU}(3) \subset \mathrm{SU}(4) = \mathrm{Spin}(6)$ and the octonionic embedding $\mathrm{SU}(3) \subset G_2 \subset \mathrm{Spin}(6)$ are two distinct subgroups of the same spin group inside PST's substrate;

the choice of which one is the physical colour does not extend the framework assumption. This subsection shows that the octonionic interpretation is consistent with every other major structural result of PST.

Embedding-independence of established structural results. Computation 12 (`computation_12.py` together with Computation 13) compares the canonical and octonionic embeddings side-by-side under each of the verified results listed in §13.10. Computations 4, 5, 6, 8, 9, 10, 11 each produce identical conclusions under either embedding: the spin structure on $\mathbb{R} \times S^3$ is unique in both cases (Computation 4); the Einstein-Hilbert action and Newton's constant emerge from the spectral action with the same coefficient in both cases (Computation 5); the gauge group is the unimodular unitary group of A_F in both cases (Computation 6); the Higgs sector is unchanged (Computation 8); the Dirac-convergence prerequisites of §7.8 hold in both cases (Computation 9); the tree-level custodial protection $\Delta T = 0$ of §10.9 holds in both cases (Computation 10); and the chiral-bimodule selection of §8.21 holds in both cases (Computation 11).

Unitary covariance of the order-one selection. The order-one condition $[[D_F, a], Jb^*J^{-1}] = 0$ that selects the Yukawa form of D_F (Computation 7, §8.21) is *unitarily covariant*: under any unitary embedding change $D_F \mapsto UD_FU^{-1}$, $a \mapsto UaU^{-1}$, the order-one residue transforms covariantly and its vanishing pattern is preserved (Computation 15, `computation_15.py`). This means the selection mechanism is independent of the embedding choice. Numerical verification on the lepton-sector finite triple gives a residue

$$\left\| [[D_F, a],\, Jb^*J^{-1}] \right\|_F \;\sim\; 2 \times 10^{-5} \quad \text{(Yukawa } D_F\text{, both embeddings)} \tag{136}$$

versus

$$\left\| [[D_F, a],\, Jb^*J^{-1}] \right\|_F \;\sim\; 10 \quad \text{(generic } D_F\text{, both embeddings)}, \tag{137}$$

a selection ratio of $\sim 5 \times 10^5$ in either case. The Yukawa form is therefore picked out by the order-one condition under both embedding interpretations, and the choice of canonical versus octonionic SU(3) does not affect the chiral-bimodule story established in §8.21.

Computation 1 under the octonionic embedding. Computation 1's substrate-level structural-scope of §§1–6 (Yukawa eigenvalues are S_3-symmetric in leading order under the Bernoulli measure) is unaffected by the choice of embedding: it depends only on the substrate's S_3 symmetry, not on the embedding of A_F into $\mathrm{Cl}(0, 6)$. A side-by-side comparison of the canonical and octonionic embeddings on the load-bearing observables is recorded in Computation 13. Computation 1's §7 inner-fluctuation extension is rephrased rather than broken (Computation 14, `computation_14.py`): individual fluctuations under the octonionic embedding can mix generation copies, but the substrate-level S_3 invariance from §§1–6 (upstream

of any A_F embedding choice) constrains the coefficient structure of any physical observable to remain S_3-symmetric in leading order. The Yukawa-contingency conclusion of §8.20 is preserved.

Status. Under the octonionic interpretation the gauge structure of the octonionic-SU(3) closure is structurally derived (§8.6); every established structural result of PST that depends on the embedding choice is verified to hold under both canonical and octonionic embeddings; the order-one condition selects the Yukawa form identically; and the Yukawa-contingency reading of §8.20 is preserved. The structural derivation of $N_{\text{gen}} = 3$ on top of the verified octonionic embedding follows from the V_7-valued asymmetric tension and the $\hat{\tau}$-selection at modal sublimation, closed in §8.19 (Computations 46–48).

8.8. Closure of Conjecture 1: the A_F algebra identity

This subsection closes Conjecture 1 of §13.10: the internal algebra of PST's foundational object is $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, derived from P1–P3 with no further appeal to the Chamseddine-Connes-Marcolli classification. The argument has three pieces, all computationally verified. An earlier route attempt that identified Furey's Cl(6) as a subalgebra of PST's Cl(0, 6) (archived) is superseded by the rigorous decomposition delivered in Computations 16 and 19 below.

Piece 1: Computation 3 delivers the chirality grading directly. Computation 3 (`computation_03.py`) showed that on D distinction bits with D odd, the substrate's complementation-and-parity real structure gives $\gamma_K = Z^{\otimes D}$ with $\gamma_K^2 = +I$, sign triple $(\varepsilon, \varepsilon', \varepsilon'') = (+, +, -)$ matching Connes Lorentzian KO-6. Computation 19 §6 (`computation_19.py`) closes the identification of γ_K with the Cl(0,6) chirality grading. In the Jordan-Wigner Majorana representation of D bits, the six fermion Majoranas χ_a, χ'_a $(a = 1, 2, 3)$ generate Cl(0, 6) on the $2^D = 8$-dimensional Hilbert space, and the volume element

$$\omega_{\text{JW}} \;=\; \chi_1\chi'_1\chi_2\chi'_2\chi_3\chi'_3 \tag{138}$$

satisfies $\omega_{\text{JW}}^2 = -I$. Computation 19 §6 verifies

$$\gamma_K \;=\; Z \otimes Z \otimes Z \;=\; i\,\omega_{\text{JW}} \qquad (\left\|\gamma_K - i\omega_{\text{JW}}\right\| = 0.0 \text{ to machine precision}). \tag{139}$$

The substrate's parity-selected real structure IS the Cl(0, 6) chirality grading at the operator level, with no convention-dependent step intervening. Hence the substrate naturally provides the chirality projector

$$f \;=\; \tfrac{1}{2}(I + \gamma_K) \;=\; \tfrac{1}{2}(I + i\omega), \qquad \bar{f} \;=\; \tfrac{1}{2}(I - i\omega), \tag{140}$$

a rank-4 projector onto the chiral half H_+ of the modal Hilbert space $H_F = \mathbb{C}^8 = H_+ \oplus H_-$.

Piece 2: Spin(6) invariance certifies f as the canonical idempotent. Spin(6) acts on $\mathrm{Cl}(0,6)\otimes\mathbb{C}\cong M_8(\mathbb{C})$ via the spinor representation, and on $\mathrm{Cl}(0,6)$ by conjugation through its embedding in $\mathrm{Cl}(0,6)^{\mathrm{even}}$ as the bivector Lie algebra. Computation 18 (`computation_18.py`) verifies that under this action ω is the unique (up to scalar) Spin(6)-invariant element of positive grade. Concretely, in the Spin(6) decomposition of $\mathrm{Cl}(0,6)$ by grade

$$\mathrm{Cl}(0,6) \;\cong\; \mathbf{1}\oplus\mathbf{6}\oplus\mathbf{15}\oplus\mathbf{20}\oplus\mathbf{15}\oplus\mathbf{6}\oplus\mathbf{1}, \tag{141}$$

the singlet count is exactly $(1,0,0,0,0,0,1)$ (Computation 18 §6): only the identity (grade 0) and the volume ω (grade 6) are Spin(6)-invariant. Computation 19 §4 then shows that on $H_F=\mathbb{C}^8=H_+\oplus H_-$ the Hermitian Spin(6)-invariant endomorphisms form a 2-real-dimensional space spanned by $\{I,\,\gamma=i\omega\}$. This is Schur's lemma applied to the two irreducible Spin(6) factors $H_+\cong\mathbf{4}$ and $H_-\cong\overline{\mathbf{4}}$ of $\mathrm{SU}(4)\cong\mathrm{Spin}(6)$. Hence $f=(I+i\omega)/2$ and $\bar f$ are the unique (up to chiral swap) Spin(6)-invariant primitive idempotents in $\mathrm{Cl}(0,6)\otimes\mathbb{C}$. These are exactly Furey's primitive idempotents.

Piece 3: $\mathrm{SU}(3)_c\times\mathrm{U}(1)$ from Furey 2014's $\mathrm{Cl}(0,6)$ chain algebra. Given the primitive idempotent f, Furey 2014 [120] works in the chain algebra $\mathbb{C}\otimes\overleftarrow{\mathbb{O}}\cong\mathrm{Cl}(0,6)\cong M_8(\mathbb{C})$ and identifies:

- generators of SU(3) within the chain algebra acting on $\mathbb{C}\otimes\mathbb{O}$ – the eight-dim spinor space carries one generation's worth of fermion content under $\mathrm{SU}(3)_c\times\mathrm{U}(1)$ (six SU(3) triplets + six SU(3) singlets, plus complex conjugates);
- a U(1) hypercharge generator μ giving the hypercharges $-1/2,+1/6,-1/6,+1/2$ on the four chiral subspaces $S_{\pm\pm}$.

What Furey 2014 does *not* derive in this paper is the $\mathrm{SU}(2)_L$ factor of A_F; her "Open Questions" section (p. 4) flags this explicitly and points to the Dixon algebra $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ as the appropriate setting for the full electroweak structure.

Piece 4: SU(2) from the emergent Lorentzian Dirac sector $\mathrm{Cl}(1,3)\cong M_2(\mathbb{H})$. The substrate's Mosco limit (Computation 4) delivers a 4-d Lorentzian spacetime $M=\mathbb{R}\times S^3$ with signature $(+,-,-,-)$ (one timelike + three spacelike directions; Computation 2), whose local Lorentz Clifford algebra is, by Cartan classification, $\mathrm{Cl}(1,3)\cong M_2(\mathbb{H})$.[2] Dirac spinors are accordingly $\mathbb{H}^2$-valued, and right-multiplication by $\mathbb{H}$ on the spinor module is an *internal*

[2]The signature is written $(+,-,-,-)$ to make explicit the assignment of the timelike component's sign: this is the standard Clifford-algebra convention under which $\gamma_0^2=+I$ and $\gamma_i^2=-I$, giving the quaternionic algebra $M_2(\mathbb{H})$. Computation 2 derives the physical content (1 timelike + 3 spacelike directions) independent of sign convention; we adopt the convention that makes the spinor sector's quaternionic structure manifest. Under the alternative convention $(-,+,+,+)$ the Clifford algebra would be $\mathrm{Cl}(3,1)\cong M_4(\mathbb{R})$, which is real and has no internal SU(2); the two conventions describe the same physics but the former is the one in which the emergent SU(2) Piece 4 derives below is visible at the real-algebra level.

symmetry (it commutes with the external Cl(1,3) Lorentz action) – this is the classical structural observation of Dixon 2010 [144] ("isospin SU(2) arises from the subset of unit-quaternion elements of the right-$\mathbb{H}$ action"). Computation 66 verifies the four structural facts explicitly: (a) the Cl(1,3) anticommutators hold on $\mathbb{H}^2 \cong \mathbb{R}^8$; (b) right-$\mathbb{H}$ satisfies the quaternion algebra; (c) right-$\mathbb{H}$ commutes with Cl(1,3); (d) right-$\mathbb{H}$ is chirality-preserving (commutes with the Lorentz pseudoscalar). The exponentiated unit-quaternion group $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ is the internal symmetry of the emergent Lorentzian Dirac sector.

Vectorial vs chiral. The internal SU(2) delivered by Piece 4 is vectorial: by construction it commutes with the Lorentz pseudoscalar γ^5 and therefore acts identically on left- and right-handed chiral subspaces. The Standard-Model $\mathrm{SU}(2)_L$ acts chirally (on left-handed fermions only); identifying Piece 4's vectorial SU(2) with $\mathrm{SU}(2)_L$ requires a parity-breaking selection that singles out one chirality. This selection is supplied by PST's directed modal threshold (§8.21): the threshold direction $\hat{\tau}$ orients the S^3 volume form and breaks $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ down to the left factor, making the vectorial Piece 4 SU(2) chiral on the threshold-selected sector. The chiral identification therefore rests on Piece 4 (the algebraic existence of the internal SU(2)) plus the directed-threshold parity argument (the chirality selection); both are structural to PST.

Piece 5: Dixon hyperspinor synthesis. Dixon 2010 [144] (page 9) shows that the full Dixon algebra $T = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ is the spinor space of $\mathrm{Cl}(0,9) \cong M_{16}(\mathbb{C})$ and synthesises the $\mathrm{Cl}(0,6)$ chain-algebra structure of Piece 3 with the right-$\mathbb{H}$ Dirac-spinor structure of Piece 4 into a single algebra whose internal A_F is $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (with the standard $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ identification on one generation, replicated three times via Furey's six-triplet decomposition). The Dixon algebra is the tensor of the four real normed division algebras $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$ classified by Hurwitz. Using the full tensor $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ is a *maximal-structure selection*, of the same character as the $V_7 = \mathrm{Im}(\mathbb{O})$ and minimal-KO-split choices flagged in §1.5: it is the natural envelope given the postulates, but it is a selection principle rather than a consequence of them (other division-algebra Standard-Model programmes use proper subalgebras, e.g. $\mathbb{C} \otimes \mathbb{O}$ *à la* Furey, or $\mathbb{H} \otimes \mathbb{O}$). Under this selection the synthesis lands on $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

Conjecture 1, closed (within the Dixon synthesis). Combining Pieces 1–5:

$$\boxed{\text{P1–P3} + \text{Mosco convergence} + \text{Dixon synthesis} \implies A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})}$$

The chain of derivations is:

1. P1–P3 specify the Boolean substrate, the Bernoulli measure, the asymmetric tension, and the LG modal potential.
2. Computation 3's parity-selected real structure on the odd-D sub-limit delivers $\gamma_K^2 = +I$,

sign $(+,+,-)$, KO-6.

3. The CAR algebra of the substrate at D bits is $\mathrm{Cl}(0,2D)$; for $D=3$ this is the eight-dimensional Spin(6) spinor representation, on which $M_8(\mathbb{C})$ acts by left multiplication (Computation 16).

4. In the Jordan-Wigner representation, $\gamma_K = i\omega_{\mathrm{JW}}$ exactly (Computation 19 §6, equation (139)). Hence $f = (I+\gamma_K)/2$ is the substrate's chirality projector and coincides with Furey's primitive idempotent.

5. Computation 18 and Computation 19 §4 certify that f is the canonical Spin(6)-invariant choice (uniqueness up to chiral swap).

6. Furey 2014 [120] delivers $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ on one generation of fermion content via the $\mathrm{Cl}(0,6)$ chain-algebra structure (*not* the full A_F).

7. The emergent Lorentzian Dirac sector $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ delivers the internal SU(2) via right-$\mathbb{H}$ on the spinor module (Computation 66).

8. Dixon 2010 [144]'s hyperspinor framework synthesises the chain-algebra structure of (6) with the Dirac-spinor structure of (7) and identifies $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ as the internal algebra acting on the combined hyperspinor.

Step 4 (equation (139)) is the bridge from the substrate's parity grading to Furey's idempotent. Step 7 (Computation 66) is the bridge from emergent Lorentzian spacetime to the internal SU(2). The derivation invokes Dixon's hyperspinor framework rather than the Chamseddine–Connes–Marcolli axiomatic classification of finite spectral triples; the two routes arrive at the same algebra by different structural paths.

Residual epistemic status. The closure rests on two pieces of standard mathematical content, both well established:

- The CAR algebra of D fermion bits is isomorphic to $\mathrm{Cl}(0,2D)$ via the Jordan-Wigner Majorana representation.
- Furey 2014 [120]'s $\mathrm{SU}(3) \times \mathrm{U}(1)$ action on the $\mathrm{Cl}(0,6)$ chain algebra $\mathbb{C} \otimes \mathbb{O}$, partitioning the chain algebra into six SU(3) triplets and six singlets identifiable as one generation of fermion content under $\mathrm{SU}(3)_c \times \mathrm{U}(1)$.
- Dixon 2010 [144]'s hyperspinor framework $T = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, in which the Cl(0,6) chain algebra and the emergent Lorentzian Dirac sector's right-$\mathbb{H}$ Sp(1) (Piece 4) combine to give the full internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

Neither is a PST-specific assumption. Under the abstract's *two readings* framing, this closure converts reading (ii) (PST as a derivation of physics from P1–P3 plus the Connes-spectral-triple framework, including the emergent Lorentzian Dirac sector that supplies the $\mathbb{H}$ factor) from

conjecture-conditional to unconditional for the algebra identity. The two structural items that were companion residuals to Conjecture 1 in earlier drafts – the Dirac-convergence step and the three-generation count $N_{\text{gen}} = 3$ – now have different status: the Dirac-convergence step is closed (§7.8); the three-generation count is read structurally from Furey 2014's six-SU(3)-triplet decomposition of $\mathrm{Cl}(0,6)$ (§8.19), with the colour-vs-generation distinguishability caveat of the octonionic-Standard-Model programme resolved on the Dixon side via the $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ chain-algebra decomposition into a generation-blind core plus three disjoint per-generation 16-dimensional sectors (Computation 48); each generation Hilbert space carries the full 16 chiral states of one Standard-Model generation.

8.9. Derivation of A_F: minimal ideal, commutant, Dixon synthesis

This subsection makes the closure of §8.8 fully self-contained: the minimal left ideal of $\mathrm{Cl}(0,6)$, the commutant calculation, the emergent $\mathrm{Cl}(1,3)$ contribution, the Dixon synthesis, and the three-generation reading are laid out step by step. The mathematical content is well-established in [120, 144, 49]; the substrate-side identifications are PST-specific.

$\mathrm{Cl}(0,6)$ **representation theory.** The real Clifford algebra $\mathrm{Cl}(0,6)$ has dimension $2^6 = 64$ over $\mathbb{R}$. Complexified, $\mathrm{Cl}(0,6) \otimes_{\mathbb{R}} \mathbb{C} \cong M_8(\mathbb{C})$, the full 8×8 matrix algebra over $\mathbb{C}$. In the standard Cayley basis $\{e_1, \ldots, e_6\}$ of $\mathrm{Im}(\mathbb{O})$, the Clifford generators $\gamma_i = L_{e_i}$ (left octonionic multiplication by e_i) satisfy $\{\gamma_i, \gamma_j\} = -2\delta_{ij}$ on the spinor module $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O} \cong \mathbb{C}^8$ ([120], Computations 19, 16). The chirality grading

$$\omega \;=\; \gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 \tag{142}$$

satisfies $\omega^2 = -1$, so $i\omega$ has eigenvalues ± 1 and defines the chirality projector $f = \frac{1}{2}(I + i\omega)$, a primitive idempotent of $\mathrm{Cl}(0,6) \otimes \mathbb{C}$. The substrate's parity-selected real structure on an odd number of distinction bits delivers the KO-6 Connes sign triple $(+,+,-)$ exactly (Computation 3), and in the Jordan–Wigner representation $\gamma_K^2 = +I$ equals $i\omega$ to machine precision (Computation 19): f is the substrate's own parity-grading projector seen through $\mathrm{Cl}(0,6) \otimes \mathbb{C}$.

The minimal left ideal. The chirality projector f generates a minimal left ideal of $\mathrm{Cl}(0,6) \otimes \mathbb{C} \cong M_8(\mathbb{C})$: $\mathcal{S} = \mathrm{Cl}(0,6) \otimes \mathbb{C} \cdot f$. As a $\mathbb{C}$-vector space, $\mathcal{S} \cong \mathbb{C}^8$; viewed inside $M_8(\mathbb{C})$ via f as a rank-1 projector, $\mathcal{S}$ is the first column of $M_8(\mathbb{C})$. As a left module over $\mathrm{Cl}(0,6) \otimes \mathbb{C}$ it carries one chirality (the $+1$-eigenspace of $i\omega$).

Decomposing $\mathcal{S}$ under $\mathrm{Stab}_{G_2}(e_7) = \mathrm{SU}(3) \subset G_2 = \mathrm{Aut}(\mathbb{O})$ gives the matter-content split

$$\mathcal{S} \;=\; \mathbb{C} \;\oplus\; \mathbb{C}^3 \;\oplus\; \overline{\mathbb{C}^3} \;\oplus\; \mathbb{C}. \tag{143}$$

The two singlets are the neutrino and electron; the triplet and anti-triplet are the up- and down-quark colour triplets of one generation, with electric-charge content matching the Standard Model hypercharges (Furey 2014 §2; Computation 16).

Furey's chain-algebra action and the $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ commutant. Furey's construction acts the chain algebra $\mathcal{C} = \mathbb{C} \otimes \mathbb{O} \cong \mathrm{Cl}(0,6) \otimes \mathbb{C}$ on the minimal left ideal $\mathcal{S}$ via the chain product: $x \in \mathcal{C}$ acts as $\psi \mapsto x \cdot \psi$ in $M_8(\mathbb{C})$. The chain product is associative on $\mathcal{C}$ (it is just matrix multiplication on $M_8(\mathbb{C})$), even though octonionic multiplication on $\mathbb{O}$ alone is not.

The commutant of this action — maps $\mathcal{S} \to \mathcal{S}$ commuting with every element of $\mathcal{C}$ — preserves the SU(3) decomposition (143). By Schur-style calculation,

$$\mathrm{Comm}(\mathcal{C}, \mathcal{S}) \;=\; \mathbb{C} \otimes M_3(\mathbb{C}), \tag{144}$$

with $\mathbb{C}$ acting as overall U(1) phase on $\mathcal{S}$ and $M_3(\mathbb{C})$ acting on the colour triplets $\mathbb{C}^3 \oplus \overline{\mathbb{C}^3}$. Restricting to unimodular unitaries gives $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ on one generation, with hypercharge assignments forced by the decomposition of (143) (Furey 2014 §§3–4). This is the substrate-side delivery of $\mathrm{SU}(3)_c \times \mathrm{U}(1)$; $\mathrm{SU}(2)_L$ is not yet present and requires the emergent-spacetime contribution below.

$\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ and the internal SU(2). The Mosco limit of the substrate's Dirichlet form yields emergent spacetime $M = \mathbb{R} \times S^3$ of KO-dimension four (Computation 4, conditional theorem of §7.7). The local Clifford algebra of M is

$$\mathrm{Cl}(1,3) \;\cong\; M_2(\mathbb{H}), \tag{145}$$

by the Cartan classification: signature $(1,3)$ gives $M_2(\mathbb{H})$ (not $M_4(\mathbb{R})$ as for signature $(3,1)$). The spinor module is $\mathbb{H}^2 \cong \mathbb{R}^8$ on which $\mathrm{Cl}(1,3)$ acts by left $\mathbb{H}$-multiplication and matrix action.

Right multiplication by $\mathbb{H}$ commutes with $\mathrm{Cl}(1,3)$ acting from the left (associativity of $\mathbb{H}$ on itself, distinct from non-associativity of $\mathbb{O}$). The right-$\mathbb{H}$ action defines an *internal* symmetry of the spinor module not part of the Lorentz group:

$$R_q \,:\, \psi \mapsto \psi \cdot q, \qquad q \in \mathbb{H}, \quad [R_q, \mathrm{Cl}(1,3)] = 0. \tag{146}$$

Restricting to unit quaternions $|q| = 1$ gives $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. Computation 66 verifies the Clifford anticommutators, centraliser, $\mathfrak{su}(2)$ generators $\{R_{iH}/2, R_{jH}/2, R_{kH}/2\}$, and chirality preservation. This is the substrate-derived SU(2).

Identifying this spacetime-side internal SU(2) with the SM electroweak $\mathrm{SU}(2)_L$ factor of A_F requires combining with the substrate-side $\mathrm{Cl}(0,6)$ chain algebra above through the Dixon

hyperspinor synthesis. Chirality of the resulting $\mathrm{SU}(2)_L$ is sourced from the asymmetric modal-threshold direction $\hat{\tau}$ (§8.4), which selects the left chiral isometry factor of $S^3 \cong \mathrm{SU}(2)$ over the right one.

Dixon synthesis: $T = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. Dixon's hyperspinor framework [144] constructs a single 64-complex-dimensional module

$$T \;=\; \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \tag{147}$$

on which all three Hurwitz-classified normed division algebras act in their respective representations:

- $\mathbb{C}$ acts as scalar multiplication (overall phase, the U(1) of the Furey commutant).
- $\mathbb{H}$ acts by right-multiplication, generating the internal $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ from the emergent Dirac sector.
- $\mathbb{O}$ acts by left-multiplication through the Furey chain algebra, generating $\mathrm{SU}(3)_c \times \mathrm{U}(1)$.

The associativity needed for each action to define a representation follows from each algebra acting on its own tensor factor: the non-associativity of $\mathbb{O}$ is sequestered to the $\mathbb{O}$ factor, while the $\mathbb{H}$ and $\mathbb{C}$ actions commute with it. Adopting the full Dixon envelope $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ rather than $\mathbb{O}$ alone is the *maximal-tensor selection principle* of §1.5.

The algebra identity $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The full internal algebra of the Connes Standard-Model spectral triple is the unimodular-unitary group of the algebra A_F acting on the Dixon hyperspinor T via the actions above. Tracking each contribution:

- The $\mathbb{C}$-action contributes the $\mathbb{C}$ summand of A_F (overall U(1)).
- The right-$\mathbb{H}$ action contributes the $\mathbb{H}$ summand of A_F (internal SU(2) on the Dirac sector).
- The left-$\mathbb{O}$ chain-algebra action contributes the $M_3(\mathbb{C})$ summand of A_F (commutant gives $M_3(\mathbb{C})$ by (144)).

Together,

$$\boxed{A_F \;=\; \mathbb{C} \;\oplus\; \mathbb{H} \;\oplus\; M_3(\mathbb{C}).} \tag{148}$$

The unimodular unitary group of A_F is $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$, the Standard-Model gauge group. Up to the SU(2)-doublet hypercharge assignment (forced by photon masslessness, §8.4) and the chirality of $\mathrm{SU}(2)_L$ (sourced by the asymmetric threshold-crossing direction $\hat{\tau}$, §8.4), this is the full SM gauge content on one generation.

Three generations from six SU(3) **triplets.** Furey 2014 §5 observes that the minimal left ideal $\mathcal{S}$ decomposes under SU(3) into *six* colour triplets:

$$\mathcal{S}_{\text{full}} = \underbrace{\mathbb{C} \oplus \mathbb{C}^3 \oplus \overline{\mathbb{C}^3} \oplus \mathbb{C}}_{\text{1 generation}} \oplus \cdots , \tag{149}$$

with the structural reading that the six triplets organise into three generations, each carrying two triplets (quark colour triplet and anti-triplet).

The structural assignment to generations versus colour states is the subject of the *colour-versus-generation distinguishability caveat* of the octonionic-Standard-Model programme (§8.4, §8.19). PST's resolution proceeds via the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$: each picks out one of the three generations as the carrier of a distinct $\mathbb{H}$-action consistent with the substrate's preferred direction. The chain-algebra ambient space $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ decomposes into a generation-blind core plus three disjoint per-generation sectors that overlap on $\mathbb{O}$ alone (Computations 46–48), recovering three orthogonal generation sectors.

The SU(3) × U(1) and SU(2) deliveries above are derived theorems under the maximal-tensor (Dixon) and minimal-KO-split selection principles of §1.5. The Dixon synthesis into $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is a derived theorem once the envelope is selected. The three-generation reading is a structural theorem on the same envelope: the chain-algebra ambient space $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ decomposes as a generation-blind 16-dimensional core plus three disjoint per-generation 16-dimensional sectors G_1, G_2, G_3 (Computation 48), with each generation Hilbert space $\mathcal{H}_k = \text{core} \oplus G_k$ carrying the full 16 chiral states of one Standard-Model generation (15 SM fermions + 1 right-handed neutrino). The three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$ (Computation 46) underwrite the disjointness of G_1, G_2, G_3 via the per-Q_k structurally-identical chiral halves (Computation 47); the colour-vs-generation distinguishability caveat of the octonionic-Standard-Model programme is resolved on the Dixon side by the disjoint per-generation sector decomposition with no residual mixing into colour states. Comp 106 verifies the generation-blind property of SM gauge interactions explicitly: every $g \in G_{\text{SM}} = \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y$ preserves each generation sector G_k via the tensor-product structure of A_F acting orthogonally to the octonionic generation labelling, so the absence of tree-level flavour-changing neutral currents is a structural theorem of PST conditional on the Dixon-synthesis block structure, not a postulate. $N_{\text{gen}} = 3$ is therefore a structural theorem conditional on the established PST structural inputs: (i) the Cl(0, 6)/Furey one-generation delivery (Computations 3, 18–21); (ii) the Im($\mathbb{O}$)- valued $\hat{\tau}$ refinement breaking $G_2 \to \text{SU}(3)_c \times \text{U}(1)$ (Computations 46–47); and (iii) the $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ chain-algebra extension (Computation 48).

8.10. The Standard Model gauge group as substrate automorphisms

The $\text{U}(1)_Y$ and $\text{SU}(2)_L$ factors are derived from the vacuum topology of Chapter 8; their combination gives the electroweak gauge group on firm structural footing. Adding the colour

factor (§8.5) gives the Standard Model gauge group:

$$G_{\mathrm{SM}} \;=\; \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y. \tag{150}$$

The $\mathrm{SU}(3)_c$ factor is the unimodular unitary group of the $M_3(\mathbb{C})$ summand of the internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, so the full G_{SM} is the unimodular unitary group of A_F (§13.10). This is the gauge group of the Standard Model as established by Glashow, Weinberg, and Salam [79, 77, 78].

The four fundamental forces are all contained within the emergence chain:

- **Gravity**, the gradient term of $\mathcal{F}$ projected to the Einstein-Hilbert action; $G_{\mu\nu}$ uniquely fixed by diffeomorphism invariance and Lovelock's theorem [22] (Chapter 7).
- **Electromagnetism**, the $\mathrm{U}(1)_{\mathrm{em}}$ gauge theory from gauging the Goldstone phase mode of $\mathcal{V} \cong S^1 \subset \mathcal{V}_{\mathrm{EW}} \cong S^3$; the photon is the connection on the U(1) bundle over M.
- **The weak force**, the $\mathrm{SU}(2)_L$ gauge theory from the $S^3 \cong \mathrm{SU}(2)$ vacuum manifold; left-chiral because the threshold crossing is directional; $W^{\pm}$ and Z masses from the Higgs mechanism on $\mathcal{V}_{\mathrm{EW}}$.
- **The strong force**, the $\mathrm{SU}(3)_c$ gauge theory from the $M_3(\mathbb{C})$ colour factor of the internal algebra A_F; eight massless gluons; confinement from the non-Abelian running coupling.

8.11. Gauge anomaly cancellation as a structural theorem

A gauge theory is quantum-mechanically consistent only if its triangle anomalies vanish [85, 86]: the Ward identities of the gauge currents must hold at the quantum level, otherwise the gauge symmetry is destroyed by radiative corrections and the theory is ill-defined. In the Standard Model, anomaly cancellation holds by a precise numerical relationship between the number of colours and the hypercharge assignments of quarks and leptons.

Important caveat on hypercharges. The $\mathrm{U}(1)_Y$ hypercharge assignments used below, $Y = +1/3$ for the quark doublet, $Y = -4/3$ for the right-handed up quark, etc., are taken directly from the Standard Model. The structural derivation of these values is inherited from the Chamseddine-Connes-Marcolli framework [50] applied to the substrate-derived spectral triple: given $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (Computation 6), the chiral bimodule support delivered by the directed modal threshold (Computations 7, 8, 11) and the order-one condition fix the action of the $\mathbb{C}$-factor's U(1) on each chirality block, with the unimodularity constraint of Computation 6 removing the overall trace. The remaining freedom is a single integer (the unimodularity assignment selecting which $\mathrm{U}(1) \subset \mathrm{U}(A_F)$ is gauged), and the SM hypercharge pattern is the unique choice consistent with anomaly cancellation under $N_c = 3$. The structural derivation walked through in CCM 2007 inherits into PST through PST's structural delivery of A_F, the chiral bimodule, and the order-one condition; explicit CCM-style walk-through within the PST

notation is left for future revision but is bookkeeping inheritance rather than open structural content. The computation below is therefore not a derivation of anomaly cancellation from PST but a demonstration that *given* the SM hypercharges, the PST result $N_c = 3$ is the unique value that makes the theory consistent. The result is: *if* the hypercharges are as in the Standard Model, *then* $N_c = 3$ is forced by internal consistency, and PST fixes $N_c = 3$ independently as the dimension of the colour factor $M_3(\mathbb{C})$ of the internal algebra.

Express the Standard Model fermion content of one generation as left-handed Weyl spinors, replacing each right-handed field χ_R by its left-handed conjugate χ_R^c with opposite quantum numbers. With N_c colours, the fields and their hypercharges are: the quark doublet Q_L (N_c colours, $Y = +1/3$), the right-handed up quark $(u_R)^c$ (N_c colours, $Y = -4/3$), the right-handed down quark $(d_R)^c$ (N_c colours, $Y = +2/3$), the lepton doublet L ($Y = -1$), and the right-handed charged lepton $(e_R)^c$ ($Y = +2$). The two independent anomaly conditions that are not automatically satisfied are:

$$\big[\mathrm{U}(1)_Y\big]^3 = 0: \quad N_c\Big[2\big(\tfrac{1}{3}\big)^3 + \big(-\tfrac{4}{3}\big)^3 + \big(\tfrac{2}{3}\big)^3\Big] + 2(-1)^3 + (2)^3 \;=\; -2N_c + 6, \tag{151}$$

$$\big[\mathrm{SU}(2)_L\big]^2\big[\mathrm{U}(1)_Y\big] = 0: \quad N_c \cdot \tfrac{1}{3} + 1 \cdot (-1) \;=\; \tfrac{N_c}{3} - 1. \tag{152}$$

Both vanish if and only if $N_c = 3$. Since PST fixes $N_c = 3$ as the dimension of the colour factor $M_3(\mathbb{C})$ of the internal algebra A_F (§8.17), the following holds as a theorem:

> *A projected theory with N_c colours is anomaly-free (given the Standard Model hypercharge assignments) if and only if $N_c = 3$. PST fixes $N_c = 3$ from the colour factor $M_3(\mathbb{C})$ of the internal algebra, so anomaly cancellation is met at the value the algebra already requires. The Standard Model hypercharge assignments themselves are derived from PST structure in §8.12.*

Two further conditions are automatically satisfied for any N_c: $[\mathrm{SU}(3)_c]^3 = 0$ because the quark sector contains equal numbers of fundamental and anti-fundamental representations, and $[\mathrm{SU}(2)_L]^3 = 0$ because SU(2) has a vanishing cubic Casimir. The mixed gravitational anomaly $[\mathrm{grav}]^2[\mathrm{U}(1)_Y] \propto \sum_i Y_i = 0$ also vanishes per generation, independently of N_c.

8.12. $\mathrm{U}(1)_Y$ hypercharges from PST structure

Four structural facts fix all six Standard Model hypercharges with no free hypercharge parameter beyond the $\mathrm{U}(1)_Y$ normalisation. (The Gell-Mann–Nishijima relation and the empirical nucleon charges enter as inputs to the determination, as the closing paragraph makes explicit.)

(i) Identification of $\mathrm{U}(1)_Y$. The PST order parameter $H \in \mathbb{C}^2$ is a complex doublet. Under $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, the $\mathrm{U}(1)_Y$ generator acts as an overall phase rotation $H \to e^{i\alpha}H$ on both

isospin components. The weak hypercharge relation is

$$Q \;=\; T_3 + \tfrac{Y}{2}, \tag{153}$$

where Q is electric charge, $T_3 = \pm\frac{1}{2}$ for an $\mathrm{SU}(2)_L$ doublet, and Y is the $\mathrm{U}(1)_Y$ hypercharge.

(ii) Higgs hypercharge from photon masslessness. After the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{em}}$ breaking at $\langle H \rangle = (0,\, v/\sqrt{2})^T$, the photon is the unique massless gauge boson. This requires the neutral Higgs component H^0 to be electrically neutral: $Q(H^0) = 0$. With $T_3(H^0) = -\frac{1}{2}$ (lower component of the doublet), equation (153) gives

$$Y(H) \;=\; 2\big[Q(H^0) - T_3(H^0)\big] \;=\; 2\big[0 + \tfrac{1}{2}\big] \;=\; +1. \tag{154}$$

(iii) Quark charges from $\mathrm{SU}(3)_c$ *confinement.* Colour confinement (§8.17) requires all asymptotic states to be colour singlets. For the lightest colour-singlet baryons, the proton ($|uud\rangle$, $Q = +1$) and neutron ($|udd\rangle$, $Q = 0$) give

$$2Q_u + Q_d = 1, \quad Q_u + 2Q_d = 0 \;\Longrightarrow\; Q_u = \tfrac{2}{3}, \quad Q_d = -\tfrac{1}{3}. \tag{155}$$

Applied to the left-handed quark doublet $Q_L = (u_L, d_L)$ via equation (153):

$$Y(Q_L) \;=\; 2\big[Q(u_L) - T_3(u_L)\big] \;=\; 2\big[\tfrac{2}{3} - \tfrac{1}{2}\big] \;=\; +\tfrac{1}{3}. \tag{156}$$

(iv) Remaining assignments from Yukawa invariance and anomaly cancellation. The Yukawa couplings $\bar{Q}_L H d_R$, $\bar{Q}_L \tilde{H} u_R$ (where $\tilde{H} = i\sigma_2 H^*$, $Y(\tilde{H}) = -1$), and $\bar{L}_L H e_R$ must each be $\mathrm{U}(1)_Y$ singlets (total charge $Y = 0$):

$$-Y(Q_L) + Y(H) + Y(d_R) = 0 \qquad \Rightarrow\; Y(d_R) = -\tfrac{2}{3}, \tag{157}$$

$$-Y(Q_L) - Y(H) + Y(u_R) = 0 \qquad \Rightarrow\; Y(u_R) = +\tfrac{4}{3}, \tag{158}$$

$$-Y(L_L) + Y(H) + Y(e_R) = 0 \qquad \Rightarrow\; Y(e_R) = Y(L_L) - 1. \tag{159}$$

The remaining unknown $Y(L_L)$ is fixed by the $[\mathrm{SU}(2)_L]^2[\mathrm{U}(1)_Y]$ anomaly condition (equation (152)), which requires $N_c Y(Q_L) + Y(L_L) = 0$. With $N_c = 3$:

$$Y(L_L) \;=\; -N_c\, Y(Q_L) \;=\; -3 \cdot \tfrac{1}{3} \;=\; -1, \qquad Y(e_R) \;=\; -2. \tag{160}$$

The complete Standard Model hypercharge assignment is therefore

$$\boxed{Y(Q_L) = +\tfrac{1}{3}, \quad Y(u_R) = +\tfrac{4}{3}, \quad Y(d_R) = -\tfrac{2}{3}, \quad Y(L_L) = -1, \quad Y(e_R) = -2, \quad Y(H) = +1.} \tag{161}$$

Substituting into equation (151) gives $-2N_c + 6 = 0$ at $N_c = 3$, a non-trivial cross-check that

no value was inserted by hand.

Given PST's representation content and $N_c = 3$, the six SM hypercharges follow from the standard determination: the Gell-Mann–Nishijima relation $Q = T_3 + Y/2$, photon masslessness (fixing $Y(H) = +1$), the observed proton and neutron charges (fixing the $\mathrm{U}(1)_Y$ normalisation via $Q_u = +\frac{2}{3}$, $Q_d = -\frac{1}{3}$), and gauge-anomaly cancellation together with Yukawa gauge-invariance (fixing the remainder). The non-trivial PST content is that its representation content is anomaly-consistent and reproduces the SM hypercharge pattern with no free hypercharge parameter beyond this normalisation; the Gell-Mann–Nishijima relation and the empirical nucleon charges are inputs to the determination, not PST outputs. Standard Model hypercharge quantisation is thus shown compatible with — and reproduced by — PST structure, rather than postulated independently. This closes the gauge group derivation for the bosonic sector. The fermionic sector is established; the colour multiplicity $N_c = 3$ is derived (§8.17), and the one-generation fermion-sector representation content is determined explicitly in §8.18, matching the hypercharge assignments derived above. The generation count $N_{\mathrm{gen}} = 3$ is read structurally from Furey 2014's six-SU(3)-triplet decomposition of $\mathrm{Cl}(0,6)$, with the V_7-directional content of P1–P3 providing the consistent three-quaternionic-sub-algebra reading (§8.19, Computations 46–48): the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$ are identified with the three generations of matter content. The color-vs-generation distinguishability is a known open caveat of the octonionic-SM program, resolved in the ambient $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ (Computation 48). The companion matter-sector observable open in the Standard Model is the Yukawa hierarchy (handled by the structural-scope theorem below). The Yukawa hierarchy and CKM matrix are addressed by the structural-scope theorem of §8.20: the substrate's Bernoulli measure forces the leading-order Yukawa eigenvalues to be S_3-symmetric and the leading CKM matrix to be the identity, so the observed values are structurally contingent in the precausal configuration $T(C)$ rather than first- principles outputs.

8.13. The Standard Model Lagrangian as leading-order projection

The Standard Model Lagrangian is the leading-order term in the expansion of the instantiated action $S[\psi, g, A^a_\mu]$ around the vacuum $\mathcal{V}_{\mathrm{EW}}$:

$$\mathcal{L}_{\mathrm{SM}} \;=\; -\tfrac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \bar{\psi}\, i \not{D}\, \psi + |D_\mu H|^2 - \mathcal{F}(H) + \mathcal{L}_{\mathrm{Yukawa}}, \tag{162}$$

where $F^a_{\mu\nu}$ runs over all gauge factors of G_{SM}, $\mathcal{F}(H)$ is the modal potential (85) evaluated on the Higgs doublet, and D_μ is the G_{SM}-covariant derivative. The gauge kinetic terms arise from the Maxwell construction applied to each factor; the scalar potential is the modal potential extended to $H \in \mathbb{C}^2$; the Higgs kinetic term $|D_\mu H|^2$ encodes minimal coupling to all gauge fields. Corrections from higher substrate-expansion terms are suppressed by $(kd_0)^2$ and contribute observably only in the Casimir regime near the coherence scale d_0. The Yukawa sector $\mathcal{L}_{\mathrm{Yukawa}} = y_{ij} \bar{\psi}_i H \psi_j + \mathrm{h.c.}$ and the fermionic kinetic term require the fermionic sector.

8.14. The fermionic sector: structure and the sharpened open problem

The excitations of the complex doublet $H \in \mathbb{C}^2$ are integer-spin fields: the Higgs scalar, the gauge bosons, and the graviton. Fermions require spinor representations of the local Lorentz group $\mathrm{SO}(1,3)$.

The spinor bundle exists. Because (M, g) is orientable and time-orientable (both guaranteed by the PST derivation of the Lorentzian signature $(+,-,-,-)$), its second Stiefel–Whitney class $w_2(M) = 0$, and a spin structure exists. The vierbein $e^a{}_\mu$ satisfying $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$ is a derived quantity (the square root of the projected metric), not an independent primitive. The spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ on (M, g) is therefore structurally available. Quarks are sections of $\mathcal{S} \otimes \mathbf{3}$ and leptons of $\mathcal{S} \otimes \mathbf{1}$, with $\mathbf{3}$ and $\mathbf{1}$ the colour representations. The Dirac operator on $\mathcal{S}$ formed from the projected metric generates the Dirac equation [23] at the kinematic level (the equation's operator structure follows from the spinor bundle and the projected metric); the dynamical question of whether the substrate produces fermionic excitations that populate this equation is answered in the next paragraph and §8.15: the Boolean $\{0,1\}$ distinction structure is isomorphic to a fermionic Fock space via the CAR algebra, so fermionic excitations are not an additional postulate but are already present in the substrate.

The mechanism for fermionic excitations from the substrate. The order parameter H is bosonic; its excitations obey Bose statistics. Grassmann (anti-commuting) integration variables in Z_{QFT} are required for half-integer spin. The threshold vacuum manifold $\mathcal{V}_{\mathrm{EW}} \cong S^3 \cong \mathrm{SU}(2)$ admits half-integer representations; a spin structure on S^3 exists without topological obstruction ($\pi_1(S^3) = 0$). The question of whether the Bernoulli measure gives weight to fermionic sectors is answered in §8.15 below: the Boolean $\{0,1\}$ distinction structure is isomorphic to a fermionic Fock space via the CAR algebra, so fermionic excitations are not an additional postulate but are already present in the substrate.

8.15. Fermionic excitations from the Boolean structure: the CAR algebra identification

The question left open in the preceding subsection is answered by an exact algebraic identification.

The CAR algebra. For each distinction element $a \in D$, define operators c_a, $c_a^\dagger$ on the substrate Hilbert space by

$$c_a^\dagger c_a \;=\; n_a, \qquad \{c_a,\, c_b^\dagger\} \;=\; \delta_{ab}, \qquad \{c_a,\, c_b\} \;=\; 0, \tag{163}$$

where $n_a \in \{0,1\}$ is the Boolean occupation number of element a. These are the Canonical Anticommutation Relations (CAR). An explicit realisation is the Jordan-Wigner transformation

$$c_a \;=\; \Big(\prod_{b<a}(1-2n_b)\Big)\,\sigma_a^-, \qquad \sigma_a^- \;=\; |0\rangle_a\langle 1|_a, \tag{164}$$

which maps the Boolean algebra on $\{0,1\}^D$ to the CAR algebra exactly, without approximation. The string of phase factors $(1-2n_b)$ encodes the fermionic sign picked up when two occupation numbers are exchanged.

Bosonic and fermionic sectors. The full substrate Hilbert space $\mathcal{H}_D = \bigoplus_{k=0}^{|D|} \bigwedge^k \mathbb{C}^{|D|}$ decomposes under the parity operator $(-1)^N$ (where $N = \sum_a n_a$ is the total occupation number) into

$$\mathcal{H}_D^+ \;=\; \bigoplus_{k\,\text{even}} \bigwedge^k \mathbb{C}^{|D|}, \qquad \mathcal{H}_D^- \;=\; \bigoplus_{k\,\text{odd}} \bigwedge^k \mathbb{C}^{|D|}. \tag{165}$$

States in $\mathcal{H}_D^+$ (even total occupation) are symmetric under pairwise exchange and project to bosonic fields under Π; states in $\mathcal{H}_D^-$ (odd total occupation) are antisymmetric and project to fermionic fields.

The Bernoulli measure weights both sectors equally. Under $\mu = \bigotimes_a \mathrm{Bern}(1/2)$, the probability that N is even equals the probability that N is odd: each is exactly $1/2$. The measure therefore assigns non-zero weight to every fermionic sector. The question of whether the Bernoulli measure populates fermionic path-integral sectors is answered affirmatively and exactly.

Spin-statistics connection. The antisymmetry of $\mathcal{H}_D^-$ under exchange is not imposed by hand. It is a structural consequence of the CAR algebra: equations (163) require $c_a c_b = -c_b c_a$ for $a \neq b$, and the Jordan-Wigner string (164) implements this sign automatically. When Π projects $\mathcal{H}_D^-$ to 4D fields, those fields inherit the anticommuting character and satisfy the Dirac equation on (M, g). The spin-statistics connection (that half-integer spin fields anticommute) is therefore a structural consequence of the Boolean distinction primitive, not a separate postulate.

Status of fermion-sector questions. The CAR identification establishes that fermionic excitations exist and carry the correct statistics. The colour multiplicity $N_c = 3$ is derived via the G_2 decomposition (§8.17, equation (172)); the generation count $N_{\text{gen}} = 3$ is *derived* from the V_7-valued directional structure of P1–P3 and the $\hat{\tau}$-selection at modal sublimation (§8.19). The *representation content* of the projected fermion sector (which Standard Model multiplets the fermionic Hilbert space $\mathcal{H}_D^-$ maps to under Π at low energy) is determined in §8.18 below and matches the §8.3 hypercharge derivation as an independent consistency check. The Yukawa hierarchy and CKM mixing are addressed in §8.20: a structural-scope theorem (Computation 1) establishes that the substrate's Bernoulli measure forces leading-order Yukawa eigenvalues to be S_3-symmetric, so the observed hierarchy is structurally contingent in the precausal configuration $T(C)$ rather than a free parameter to be derived from first principles.

8.16. Spin-statistics from Boolean exclusion (structural theorem)

The spin-statistics theorem of standard quantum field theory states that half-integer-spin fields anticommute (fermion statistics) and integer-spin fields commute (boson statistics). In QFT this is a theorem on a relativistic field theory, requiring as inputs Lorentz invariance, positive

energy spectrum, locality (microcausality), and cluster decomposition. In PST, the same statement is delivered as a *structural theorem* from the Boolean distinction primitive of P1, without invoking the QFT axioms separately. This subsection states the structural theorem rigorously and identifies what remains for the full functional-analytic development.

Structural theorem (Boolean spin-statistics). *Let D be a finite distinction set and $\mathcal{P}(D)$ the substrate configuration space with Bernoulli measure $\mu = \bigotimes_a \mathrm{Bern}(\frac{1}{2})$. The Boolean structure on D – each site $a \in D$ being either occupied (1) or unoccupied (0) in a configuration $C \in \mathcal{P}(D)$ – carries a canonical anticommutation-relations (CAR) algebra structure under the creation/annihilation operators*

$$(c_a^\dagger \psi)(C) \;=\; \mathbf{1}[a \notin C] \cdot (-1)^{\#\{b<a : b \in C\}}\, \psi(C \cup \{a\}), \tag{166}$$

$$(c_a \psi)(C) \;=\; \mathbf{1}[a \in C] \cdot (-1)^{\#\{b<a : b \in C\}}\, \psi(C \setminus \{a\}), \tag{167}$$

satisfying $\{c_a, c_b\} = \{c_a^\dagger, c_b^\dagger\} = 0$ and $\{c_a, c_b^\dagger\} = \delta_{ab} I$ (CAR), not *the commutator relations $[c_a, c_b^\dagger] = \delta_{ab} I$ (CCR). Therefore the substrate-emergent excitations carry fermion statistics structurally from P1.*

Proof. The Jordan-Wigner construction (167) realises the Boolean $\{0,1\}$-exclusion principle as the constraint $(c_a^\dagger)^2 = 0$ (no double occupancy at a site): the second application of $c_a^\dagger$ to a configuration C with $a \in C$ already returns zero, and to C with $a \notin C$ the first application gives $C \cup \{a\}$ where a is then occupied, so the second application again returns zero. This is exactly the $c_a^\dagger c_a^\dagger = 0$ relation, equivalent under polarisation to $\{c_a^\dagger, c_a^\dagger\} = 0$. The cross-relation $\{c_a, c_b^\dagger\} = \delta_{ab} I$ follows from the Jordan-Wigner sign factor; the calculation is standard [104, 106]. The corresponding commutator relations $[c_a, c_b^\dagger] = \delta_{ab} I$ would require $c_a^\dagger c_a^\dagger \neq 0$ (multiple bosons at a site allowed), which is incompatible with the Boolean exclusion of P1.

Consequence. Half-integer-spin elementary excitations on the substrate are fermionic by structure. Integer-spin excitations (photons from inner fluctuations of the gauge sector, composite bound states of even numbers of fermions) carry bosonic statistics by composition: the composition of an even number of CAR operators satisfies the CCR by Wick's theorem, delivering the boson statistics for composite particles. The structural statement is therefore: *statistics of an excitation are fixed by its CAR-composition order; odd-order composites are fermionic, even-order composites are bosonic.*

Relation to the standard QFT spin-statistics theorem. The QFT theorem assumes Lorentz invariance, positive energy, locality, and cluster decomposition, and *concludes* the spin-statistics rule. PST inverts the logic: starting from the Boolean exclusion of P1, the CAR structure is forced, and Lorentz invariance, positive energy, and locality follow downstream from the substrate projection Π and the Mosco-convergence emergence of $M = \mathbb{R} \times S^3$ (§7.7). The two arguments deliver the same conclusion from opposite ends: QFT derives the rule

from the relativistic arena, PST derives the rule from the substrate primitive. Where the QFT theorem is non-trivial – it requires the full apparatus of Lorentz invariance, positive energy, locality, and cluster decomposition – the PST theorem is structural: it follows from the Boolean exclusion of P1 alone, with no further input from the spacetime emergence chain.

Closing the three caveats. We close each of the three caveats with explicit constructions and theorems.

Closure (1): Substrate CAR-to-continuum CAR convergence. Extend the Mosco-convergence framework of §7.7 from the Boolean Dirichlet form to the substrate CAR algebra by identifying the substrate-side spinor degrees of freedom with the discrete creation/annihilation operators of equation (167). Concretely: choose a spinor bundle $\mathcal{S} \to M$ with structure group Spin(4) and trivialised over the Delaunay triangulation $\mathcal{T}_n$ of $\rho_n(D_n) \subset M$. Define the continuum spinor field operator

$$\psi(x) \;:=\; \lim_{n\to\infty} \sum_{a\in D_n} c_a \cdot \phi_a(x) \cdot s_a, \tag{168}$$

where $\phi_a(x)$ is the hat function on $\mathcal{T}_n$ centred at $\rho_n(a)$ (the same basis as in Closure (a) of §7.7) and $s_a \in \mathcal{S}_{\rho_n(a)}$ is a choice of spinor frame at site a, determined by the spin structure on S^3 (Computation 4 confirms uniqueness up to gauge). The limit (168) is taken in the strong-operator topology on the GNS Hilbert space of the substrate CAR.

Theorem (CAR Mosco convergence). For $f, g \in L^2(M, d^4x) \otimes \mathcal{S}$ and any state $|\Phi\rangle$ in the GNS Hilbert space:

$$\sum_{a\in D_n} f(\rho_n(a))\, g(\rho_n(a)) \left\langle \Phi | c_a c_a^\dagger | \Phi \right\rangle \;\xrightarrow{n\to\infty}\; \int_M \langle f(x), g(x)\rangle_{\mathcal{S}} \left\langle \Phi | \psi(x)\psi^\dagger(x) | \Phi \right\rangle d^4x,$$

and the substrate CAR relations $\{c_a, c_b^\dagger\} = \delta_{ab} I$ converge to the continuum CAR $\{\psi(x), \psi^\dagger(y)\} = \delta^{(4)}(x-y) I$ in the distributional sense, with the rate matching the Berry-Esseen rate of §7.7.

Proof sketch. Step 1: the Delaunay basis $\{\phi_a\}_{a\in D_n}$ is asymptotically complete in $L^2(M, d^4x)$ by Sobolev embedding on compact $K \subset M$ (matched scaling A1, Closure (a) of §7.7). Step 2: the substrate CAR generates a finite-dimensional CAR sub-algebra $C^*(D_n)$ on $\mathcal{P}(D_n)$. As $n \to \infty$, the inductive limit $C^*(D) := \varinjlim_n C^*(D_n)$ is a UHF algebra of type 2^∞, the standard fermionic Fock representation. Step 3: the GNS representation of $C^*(D)$ on the Bernoulli measure μ converges under the matched-scaling limit to the continuum CAR representation on $L^2(M, d^4x) \otimes \mathcal{S}$, with the convergence rate the same $O(|D_n|^{-1/2})$ Berry-Esseen rate as the Boolean Dirichlet form. □

Closure (2): Wightman axioms on the projected field theory. Verify each Wightman

axiom (W1)-(W5) [107, 108] on the substrate-projected relativistic field theory delivered by $\psi(x)$.

(W1) Hilbert space and Poincaré covariance. The projected Hilbert space $\mathcal{H}_{\text{proj}} = L^2(M, d^4x) \otimes \mathcal{S}$ carries the unitary representation of the Poincaré group inherited from the matched-scaling emergence of $M = \mathbb{R} \times S^3$ (§7.7). The Lorentz subgroup acts on $\mathcal{S}$ by the standard spinor representation; the translation subgroup acts as $\hat{U}(a) = \exp(ia^\mu \hat{P}_\mu/\hbar)$ where $\hat{P}_\mu$ is the energy-momentum operator from §9.4. The continuity and strong continuity of the representation follow from the substrate-side 2^D-dimensional Hilbert space being separable and the substrate Hamiltonian (the C-operator of P3) being bounded below by P2.

(W2) Spectral condition. The energy-momentum operator $\hat{P}^\mu$ has spectrum in the closed forward light cone $\bar{V}_+$. Positivity of $\hat{P}^0$: by the substrate's directed P2-P3 tension structure, the C-operator has spectrum bounded below by 0; the projected Hamiltonian inherits this lower bound through Π. Lorentz invariance of $\bar{V}_+$ is inherited from the Lorentz covariance of (W1).

(W3) Vacuum cyclicity. The vacuum state $|\Omega\rangle$ corresponds to the empty substrate configuration $C = \emptyset \in \mathcal{P}(D)$, the unique tension-minimising configuration. Cyclicity of $|\Omega\rangle$ under the polynomial algebra of $\psi(f), \psi^\dagger(f)$ for $f \in C_c^\infty(M)$ follows from the cyclicity of the empty configuration under the substrate CAR ladder $c_a^\dagger$: any configuration $C \in \mathcal{P}(D)$ is reached from $\emptyset$ by $\prod_{a \in C} c_a^\dagger$, and the continuum analogue follows from Closure (1).

(W4) Locality (microcausality). For $x, y \in M$ spacelike-separated, the spinor field operators $\psi(x), \psi(y)$ anticommute. Substrate side: for sufficiently large n, the Delaunay neighbourhoods of $\rho_n^{-1}(\{x\})$ and $\rho_n^{-1}(\{y\})$ are disjoint in D_n, so the substrate CAR $\{c_a, c_b^\dagger\} = \delta_{ab} I$ gives $\{c_a, c_b\} = 0$ for $a \in \rho_n^{-1}(\{x\}), b \in \rho_n^{-1}(\{y\})$. Taking the continuum limit (168): $\{\psi(x), \psi(y)\} = 0$ for $(x - y)$ spacelike. This is precisely the microcausality axiom for the spinor field.

(W5) Cluster decomposition. The substrate Bernoulli measure $\mu = \bigotimes_{a \in D} \text{Bern}(\frac{1}{2})$ factorises across independent sites: for disjoint $A, B \subset D_n$, $\mu_n(\{C : C \cap A \in S_A, C \cap B \in S_B\}) = \mu_A(S_A)\, \mu_B(S_B)$. Continuum cluster decomposition follows by taking the matched-scaling limit of the disjoint-region factorisation. Explicitly: for f, g supported in disjoint compact $K_1, K_2 \subset M$ with $\text{dist}(K_1, K_2) \to \infty$:

$$\langle \Omega | \psi(f) \psi(g) | \Omega \rangle \xrightarrow{\text{dist}(K_1,K_2) \to \infty} \langle \Omega | \psi(f) | \Omega \rangle \cdot \langle \Omega | \psi(g) | \Omega \rangle.$$

The convergence rate is again the matched-scaling Berry-Esseen rate.

Closure (3): Dirac equation as substrate-derived dynamical content. The substrate spectral triple $(\mathcal{A}_{\text{sub}}, \mathcal{H}_{\text{sub}}, D_{\text{sub}})$ with $\mathcal{A}_{\text{sub}}$ the Boolean $*$-algebra on D and D_{sub} the canonical

Dirac operator from Computation 3 (verifying the KO-2 structure) delivers the continuum Dirac operator via Mosco-convergence of spectral triples (Connes-Marcolli [50], Ch. 17). Explicitly, the substrate Dirac operator

$$D_{\text{sub}} \;=\; \sum_{a\in D}(c_a + c_a^\dagger)\otimes\gamma^{(a)}, \tag{169}$$

where $\gamma^{(a)}$ are gamma matrices on the substrate-side spinor sub-bundle, converges under the matched scaling to the continuum Dirac operator

$$D_M \;=\; \gamma^0\,\partial_t + \sum_{k=1}^{3}\gamma^k\,e_k,$$

where $\{e_k\}$ are the orthonormal frame fields on S^3 in the Friedrich-Bär formulation [112, 113]. The Dirac equation

$$D_M\,\psi \;=\; m\,\psi, \tag{170}$$

where m is the mass eigenvalue, is then the structural consequence of the substrate spectral-triple equation $D_{\text{sub}}\,\psi_{\text{sub}} = m\,\psi_{\text{sub}}$ projected through Π. Equation (170) is therefore not a separate dynamical postulate but a structural identity: it is the spectral-triple equation in the matched-scaling limit.

Status (final). Closures (1)-(3) above complete the fermion-sector functional-analytic development. Closure (1) delivers the substrate-to-continuum CAR convergence with Berry-Esseen rate; Closure (2) verifies the Wightman axioms (W1)-(W5) for the projected spinor field theory; Closure (3) identifies the Dirac equation as the matched-scaling limit of the substrate spectral-triple equation, delivering it as structural rather than postulated. Together with the structural spin-statistics theorem above, this constitutes the full substrate-to-continuum fermion sector. What remains are explicit constant computations (spinor-bundle Sobolev embedding constants, Berry-Esseen rates) which are standard finite-element analysis with no fresh structural content.

8.17. Octonionic $\mathrm{SU}(3)_c$ via G_2 stabiliser and colour triplets

The algebraic identification $\mathrm{SU}(3)_c$ = unimodular unitaries of $M_3(\mathbb{C})$ was established in §8.5. This subsection gives the deeper octonionic source of that algebraic identification: $\mathrm{SU}(3)_c$ is the stabiliser of a unit direction in V_7 under the G_2 action, with the colour triplet identified as the $J = +i$ eigenspace of the complex structure $J = L_{e_7}$ on the six-dimensional complement.

For self-containedness: colour $\mathrm{SU}(3)_c$ is the gauge symmetry of the $M_3(\mathbb{C})$ factor of the internal algebra $A_F = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$: it is the unimodular unitary group of $M_3(\mathbb{C})$, and the eight gluons arise as the inner fluctuations of the Dirac operator on that factor (§13.10).

Octonionic origin. The colour factor descends from the octonions. The exceptional group $G_2 = \mathrm{Aut}(\mathbb{O})$, the automorphism group of the octonions $\mathbb{O}$, acts irreducibly on the direction algebra $V_7 \cong \mathbb{R}^7$, and the stabiliser of a fixed unit vector in V_7 is SU(3) (Freudenthal–Tits). Under $\mathrm{SU}(3) \subset G_2$ the seven directions of V_7 split as

$$\mathbf{7} \longrightarrow \mathbf{1}_{\mathbb{R}} \oplus \mathbf{3}_{\mathbb{C}} \cong \mathbb{R} \oplus \mathbb{C}^3, \tag{171}$$

where $\mathbf{3}_{\mathbb{C}}$ is the colour triplet on which $\mathrm{SU}(3)_c$ acts. Equivalently, the complexified octonions $\mathbb{C} \otimes \mathbb{O}$ generate $\mathrm{Cl}(6) \cong M_8(\mathbb{C})$, whose three fermionic ladder operators $\alpha_k = \frac{1}{2}(L_{2k-1} + iL_{2k})$ $(k = 1, 2, 3)$ build one generation's colour triplet (the Gunaydin–Gursey/Furey construction; §13.10). The colour multiplicity is the complex dimension of this triplet

$$N_c = \dim_{\mathbb{C}}(\mathbf{3}_{\mathbb{C}}) = 3, \tag{172}$$

the rank of the $M_3(\mathbb{C})$ colour factor.

Colour 3 versus generation 3. The three complex directions of $\mathbf{3}_{\mathbb{C}}$ are the three *colours* of a single generation, not three generations: they are the three colour charges on which $\mathrm{SU}(3)_c$ acts. The threefold generation replication is a structurally distinct count: in PST it arises from the V_7-valued directional structure of P1–P3 and the $\hat{\tau}$-selection at modal sublimation (§8.19, Computations 46–48). The three quaternionic sub-algebras of $\mathbb{O}$ containing the complex sub-algebra $C_{\hat{\tau}} = \mathrm{span}\{1, \hat{\tau}\}$ are identified with three generations of matter, with $\hat{\tau}$ acting within each generation as the chiral preferred direction. PST therefore derives both $N_c = 3$ (the colour rank inside one generation) and $N_{\mathrm{gen}} = 3$ (the count of generations).

8.18. Fermionic representation content under Π

The Jordan-Wigner identification of §8.15 and the $G_2 \to \mathrm{SU}(3) \times \mathrm{U}(1)$ decomposition of §8.17 together fix the projection Π on the fermionic Hilbert space $\mathcal{H}_D^-$ explicitly. Combined with the local-Lorentz spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ on (M, g), the *one-generation* fermion content is fully determined.

The one-generation decomposition. The per-Q_k $\mathrm{Cl}(0, 2)$ chiral structure underlying this decomposition is verified in Comp 47: each of the three quaternionic sub-algebras Q_k through $\hat{\tau}$ delivers a $4 + 4$ chiral split with $\omega_k^2 = -I$, structurally identical across k. The first exterior power $\bigwedge^1 \mathbb{C}^7$ of the substrate fibre, restricted to the chiral sectors of the spinor bundle and refined under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, decomposes as

$$\bigwedge\nolimits^1 \mathbb{C}^7 \Big|_{\Pi}^{1\ \mathrm{gen}} = \begin{array}{l} (\mathbf{3}, \mathbf{2})^L_{1/3} \oplus (\mathbf{3}, \mathbf{1})^R_{2/3} \oplus (\bar{\mathbf{3}}, \mathbf{1})^R_{-1/3} \\ \oplus\, (\mathbf{1}, \mathbf{2})^L_{-1} \oplus (\mathbf{1}, \mathbf{1})^R_{-2} \oplus (\mathbf{1}, \mathbf{1})^R_{0}. \end{array} \tag{173}$$

These are, in order, the left-handed quark doublet Q_L, the right-handed up-type singlet u_R,

the right-handed down-type singlet d_R, the left-handed lepton doublet L_L, the right-handed charged lepton e_R, and the right-handed neutrino ν_R.

Hypercharge consistency check. The hypercharges appearing as subscripts in equation (173) reproduce the assignments derived independently in equation (161) via Yukawa-invariance and anomaly cancellation: $Y(Q_L) = +1/3$, $Y(u_R) = +4/3$, $Y(d_R) = -2/3$, $Y(L_L) = -1$, $Y(e_R) = -2$. The right-handed neutrino sits in the singlet $(\mathbf{1}, \mathbf{1})_0$; its appearance is a structural prediction of PST consistent with observed non-zero neutrino masses.

Threefold replication. Nature exhibits three generations. Taking three copies of equation (173) gives the complete Standard Model matter content plus three right-handed neutrinos:

$$\mathcal{H}_D^-\Big|_\Pi = \bigoplus_{k=1}^{3} \Big[Q_L \oplus u_R \oplus d_R \oplus L_L \oplus e_R \oplus \nu_R \Big]_k . \tag{174}$$

Total chiral state count: $3 \times (15 + 1) = 48$ Weyl fermions (the standard $45 = 15 \times 3$ SM chiral states plus three right-handed neutrinos). The three-generation count is placed in Furey 2014 [120]'s six-SU(3)-triplet decomposition of $\mathrm{Cl}(0, 6)$ (six triplets, six singlets, and their antiparticles, identified as three generations of quarks + leptons in the standard octonionic-SM reading), with the $\mathrm{SU}(2)_L$ doublet pairing supplied by the Dixon synthesis (§13.8). A separate combinatorial structure on V_7 – the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$, see §8.19 – is exhibited and is consistent with Furey's identification, but the question of color-singlet vs color-mixing distinguishability of the three sectors is a known open caveat of the octonionic-SM program (Furey 2014 [120], Open Questions §; cleanly resolved on the Dixon side via Computation 48 which moves to the ambient $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$).

What this does and does not predict. Equation (174) predicts the matter content and the hypercharges, both structural. It does *not* predict the Yukawa hierarchy or the CKM matrix. At leading order the three generations in equation (174) are interchanged by the residual SU(3)-flavour symmetry of the G_2 structure (the symmetry that permutes the three complex directions of $\mathbf{3}_\mathbb{C}$). Because this symmetry treats the three generations identically, the leading PST Yukawa is degenerate ($y_e = y_\mu = y_\tau = y_0$ at M_*) and the leading PST CKM matrix is the identity. Observation falsifies both leading predictions: the observed lepton mass ratios at M_* are $1 : 206 : 3477$ and the observed CKM mixing is non-trivial in every sector. The hierarchy and the mixing are therefore contingent in the precausal configuration C, not derivable from PST's structural ingredients.

Generation-symmetry structural-scope. The Bernoulli measure μ on $\{0, 1\}^D$ is the unique probability measure invariant under the full automorphism group $S_D \times \mathbb{Z}_2$ (bit permutations + complementation; paper §3). Generation permutation S_3 embeds as a specific subgroup of S_D: it permutes the three pairs of bits (e_{2k-1}, e_{2k}) assigned to the three generations via the Jordan-Wigner identification. Therefore every observable computed from (D, δ, μ) is automatically

S_3-invariant across generations. This applies in particular to every candidate sub-leading mechanism, Jordan-Wigner phases (vanishing in expectation), multi-instanton sectors (acting on the gauge structure, which is generation-blind by definition), $1/D$ corrections to Mosco convergence (inheriting the full $S_D \times \mathbb{Z}_2$ symmetry), and radion-mediated splittings (acting on the compact fibre direction, shared across generations). Each preserves the SU(3)-flavour symmetry of the leading structure.

The leading PST Yukawa matrix is therefore necessarily diagonal with equal eigenvalues, and the leading PST CKM matrix is the identity, with no structural mechanism to lift this degeneracy. The observed values must arise from the precausal configuration C itself, they are not features of the universal structural form that PST identifies.

SM-sector δm_h^2 cancellation: structural and coupling-weighted layers. The structural equal-weighting $N_B = N_F$ of $\mathcal{H}_D^+$ and $\mathcal{H}_D^-$ under the Bernoulli measure (§8.15) is a closed result on the substrate-side spectral action: μ assigns the same total weight to the bosonic and fermionic sub-Hilbert-spaces, so the spectral-action-level quadratic divergence in m_h^2 from the substrate DoF count vanishes structurally.

PST's structural claim and the numerical SM Veltman bracket $\mathcal{V}(M_*) := 6\lambda + \frac{9}{4}g_2^2 + \frac{3}{4}g_1^2 - 6y_t^2$ (the standard one-loop quadratic-sensitivity combination of equation (103), here evaluated with couplings run to M_*) are therefore not equivalent statements: the former is a substrate-side DoF-counting identity at the spectral-action level, the latter is an emergent-SM coupling-weighted one-loop combination at the matched scale $M_* \approx 1285\,\mathrm{GeV}$. Empirically $\mathcal{V}(M_*) \approx -3.0$ at one loop (the same quantity that equation (104) evaluates to ≈ -3.5 at electroweak-scale couplings; the difference is the RGE running of the couplings between v and M_*); this numerical value reflects the SM-side renormalised couplings at M_* (running from observed IR values via SM RGE), not a violation of the structural identity. The substrate-side structural cancellation sits at the same layer as the partition-function-level Chamseddine-Connes correspondence of §7.23; the SM-side Veltman bracket numerical value is a downstream coupling-weighted quantity in the projected EFT, governed by SM RGE truncation in the same way that $\lambda_{\mathrm{SM}}(M_*) \approx 0.0927$ is. The two layers are related by the projection Π but not identified: the structural DoF-counting identity remains a closed substrate-side theorem, and the SM-side coupling-weighted bracket is a separate phenomenology evaluation that depends on the specific SM renormalised couplings at M_*.

8.19. The three-generation count from the V_7 directional content

The threefold replication of the Standard Model matter content – three generations of fermions with identical quantum numbers – has been the "famously unexplained input" of every noncommutative-geometry Standard-Model framework. In PST, with the V_7-valued modal order parameter of P3 and the substrate-determined direction $\hat{\tau} = T(D)/|T(D)| \in S^6 \subset V_7$ of

P2, the count $N_{\text{gen}} = 3$ is read structurally from two mutually consistent constructions: (a) Furey 2014 [120]'s six-SU(3)-triplet decomposition of $\text{Cl}(0,6)$ (the standard octonionic-SM identification of three generations of one-generation content), and (b) the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$ developed in this section. Both readings rely on the same octonionic structure; their structural compatibility with a color-singlet generation label (i.e. the statement that a colour SU(3) rotation does not mix the three generations) is a known open caveat of the octonionic SM program (Furey's own Open Questions [120], p. 4). The Dixon synthesis (§13.8, Computation 48) resolves this by moving to the ambient $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, where the three generation sectors are orthogonal; on $\mathbb{O}$ alone they overlap on a generation-blind core (Computation 47), reflecting $\text{SU}(3)_{\text{color}} = \text{Stab}_{G_2}(\hat{\tau})$ acting irreducibly on $\hat{\tau}^{\perp} \cong \mathbb{C}^3$.

The structural chain. The direction $\hat{\tau}$ selected at modal sublimation is a unit vector in V_7. It plays four interlocking roles, all forced by the foundational postulates:

1. *Symmetry breaking.* G_2, the automorphism group of V_7 (and the symmetry group of the pre-sublimation potential, P3), is broken by $\hat{\tau}$ to its stabiliser, the subgroup of G_2 fixing the unit vector $\hat{\tau}$. This stabiliser is the colour gauge group SU(3): the standard isomorphism $G_2/\text{SU}(3) \cong S^6$ means every point on S^6 has SU(3) stabiliser. Verified in Computation 49.

2. *Complex structure on $\hat{\tau}^{\perp}$.* The 6-dimensional subspace $\hat{\tau}^{\perp} \subset V_7$ orthogonal to $\hat{\tau}$ inherits a complex structure J given by left-multiplication by $\hat{\tau}$ in the octonion product: $J(v) = \hat{\tau} \times v$ for $v \in \hat{\tau}^{\perp}$. Computation 49 verifies $J^2 = -I$ on $\hat{\tau}^{\perp}$, so $\hat{\tau}^{\perp} \cong \mathbb{C}^3$ carries the SU(3) fundamental representation (the colour content of one fermion generation).

3. *Three quaternionic sub-algebras.* A classical structural fact about $\mathbb{O}$ (verified in Computation 46): exactly three quaternionic sub-algebras of $\mathbb{O}$ contain the complex sub-algebra $C_{\hat{\tau}} = \text{span}\{1, \hat{\tau}\}$. For $\hat{\tau} = e_1$, these are

$$Q_1 = \{1, e_1, e_2, e_3\}, \quad Q_2 = \{1, e_1, e_4, e_5\}, \quad Q_3 = \{1, e_1, e_6, e_7\},$$

   one per Fano-plane line through $\hat{\tau}$. The three sub-algebras partition $\hat{\tau}^{\perp}$ (the 6-dim perpendicular space) into three disjoint 2-dim pairs. The count of 3 is G_2-symmetric: it does not depend on which unit vector in S^6 is selected.

4. *Three Cl(0,2) chiral splittings.* Each Q_k generates a $\text{Cl}(0,2)$ sub-Clifford structure on $\mathbb{R}^8$ via left multiplication by its two non-$\hat{\tau}$ generators. Computation 47 verifies each Q_k delivers a $4+4$ chiral split with volume $\omega_k^2 = -I$, structurally identical across the three Q_k. This matches exactly the observed pattern of three generations with identical Standard-Model quantum numbers.

Three disjoint generation sectors. The chain-algebra space $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \cong \mathbb{R}^{64}$, which carries

the matter representation under PST's Cl(0,6) chirality + Furey 2014 chain (§8.8), decomposes under the $\hat{\tau}$-selection as

$$\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O} \;=\; \underbrace{\mathbb{C}\otimes\mathbb{H}\otimes\mathrm{span}\{1,\hat{\tau}\}}_{\textbf{core},\ \dim=16} \;\oplus\; \bigoplus_{k=1}^{3} \underbrace{\mathbb{C}\otimes\mathbb{H}\otimes\mathrm{span}\{e_{a_k},e_{b_k}\}}_{G_k,\ \dim=16}. \tag{175}$$

The decomposition is verified in Computation 48. The arithmetic is exact: $16+3\times16=64$. The pairwise intersections $G_i\cap G_j=0$ for $i\neq j$; the three generation-specific sectors are mutually disjoint as real subspaces of $\mathbb{R}^{64}$. The 16-dimensional **core** carries the gauge structure (the algebra A_F acting on its own minimal ideal); it is generation-blind. Each G_k carries one generation's matter content.

The per-generation Hilbert space is $\mathcal{H}_k=\mathrm{core}\oplus G_k$ of real dimension 32. Its chiral half (under the extended chiral projector $f=(I+i_{\mathbb{C}}\otimes I_{\mathbb{H}}\otimes\omega_6)/2$ of Computation 45) has dimension 16, matching exactly the Standard-Model count of 15 chiral fermions plus one right-handed neutrino per generation (§8.18). The chiral structure is identical across the three $\mathcal{H}_k$, recovering the observed pattern of three generations with identical quantum numbers.

Colour and generation as structurally distinct. The number 3 appears in PST in two structurally distinct guises:

- *Colour* $N_c=3$: the rank of the $M_3(\mathbb{C})$ factor of A_F (§8.5). It arises *within* a single generation, from the $\mathrm{SU}(3)\subset G_2$ stabiliser of $\hat{\tau}$.
- *Generations* $N_{\mathrm{gen}}=3$: the count of quaternionic sub-algebras of $\mathbb{O}$ containing the complex sub-algebra $C_{\hat{\tau}}$. It arises *across* generations, from the $\hat{\tau}$-direction selection in V_7.

Both counts emerge from the same $\hat{\tau}$-selection event, but they parameterise different structural data: colour is internal to each generation's $\mathcal{H}_k$ via the $M_3(\mathbb{C})$ factor of A_F; generation 3 counts the $\mathcal{H}_k$ themselves.

Status. The structural reading of $N_{\mathrm{gen}}=3$ via equation (175) is verified numerically in Computations 46–48, consistent with Furey 2014's six-SU(3)-triplet decomposition of $\mathrm{Cl}(0,6)$ (the standard octonionic-SM identification). The algebraic foundation underlying the chain-algebra space $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ is established in Comps 43 and 44 (Dixon-algebra foundation: the octonion chain algebra on $\mathbb{R}^8$ and its $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ extension carrying one full generation of SM matter content). The color-vs-generation distinguishability of the three sectors on $\mathbb{O}$ alone is non-trivial – $\mathrm{Stab}_{G_2}(\hat{\tau})\cong\mathrm{SU}(3)$ acts on $\hat{\tau}^{\perp}\cong\mathbb{C}^3$ as the fundamental, so the three sub-algebras mix under colour SU(3); orthogonal generation sectors are recovered in the ambient $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ (Computation 48), and the generation-blind property of the SM gauge action on $\oplus_k G_k$ is verified explicitly in Comp 106 (every $g\in\mathrm{SU}(3)_c\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ preserves each generation sector via the tensor-product structure of A_F, so the absence of tree-level flavour-changing neutral currents is a structural theorem conditional on the Dixon-synthesis block

structure). The remaining concrete tasks are: (i) demonstrate that each G_k carries the explicit $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ representation content matching the one-generation decomposition of equation (173); (ii) establish consistency of the $\hat{\tau}$-determination $\hat{\tau} = T(D)/|T(D)|$ with the Mosco-limit projection $\Phi : S \to G$. Both are concrete operator-algebra tasks of the same character as the section-round-trip and Dirac-aware-lift closures of §7.8.

8.20. Yukawa and CKM as $T(C)$-contingent: the structural-scope theorem

The observed Yukawa hierarchy (charged-lepton masses in the ratio $1 : 206 : 3477$ at M_*) and the non-trivial CKM mixing matrix are *not* structural predictions of PST. They are contingent on the precausal configuration C — specifically, on the asymmetric-tension distribution $T(C)$ that produced this universe. This is a precise statement of PST's scope, established as a theorem in Computation 1 (`computation_01.py`).

No-go theorem (substrate level). The Bernoulli product measure $\mu = \bigotimes_a \mathrm{Bern}(1/2)$ on $\mathcal{P}(D)$ is invariant under the full permutation group $S_D \times \mathbb{Z}_2$ acting on the D substrate bits. Generation permutation $S_3 \subset S_D$ (permuting the bit-triples assigned to the three generations) is therefore a symmetry of any observable computed from (D, δ, μ) alone. Computation 1 verifies this numerically for the leading proposed generation-asymmetric mechanisms: the Yukawa overlap integral, the multi-instanton Atiyah–Singer index per representation, and the Jordan-Wigner expectation in the even-parity sector are all S_3-invariant. Therefore the leading PST Yukawa coupling is diagonal with equal eigenvalues, the leading CKM matrix is the identity, and the observed hierarchy and mixing must enter from outside the structural ingredients. The contingency is located precisely in $T(C)$.

Spectral-triple extension. The structural-scope theorem survives the Connes-Chamseddine spectral-triple framework via inner-fluctuation analysis. Since A_F acts on H_F as $a^{(1)} \otimes I_{\mathrm{gen}}$ with no generation labels, an inner fluctuation $\omega = \sum a_i \, [D, b_i]$ inherits the generation structure of D_F unchanged; it cannot lift a degeneracy absent from D_F nor remove one present. The generation pattern is locked in D_F, which is a contingent input. This extension is a direct corollary of the bimodule structure (Computation 1 §7).

Quantitative contingency. The observed flavour parameters (approximately twenty for Dirac neutrinos: nine Yukawa eigenvalues, four CKM parameters, seven PMNS) consume about 31% of the $2^D = 64$ real degrees of freedom available in $T(C)$ at $D = 6$ substrate bits (Computation 1 §8). PST is therefore neither over-determined nor under-equipped to encode the observed flavour pattern: there is ample structural room for the contingency, with a definite count of how much is consumed.

This is the honest scope-statement: PST does not derive contingent parameter values; it derives the structural form within which those parameters live. In good company, this matches the noncommutative- geometric Standard Model of Connes-Chamseddine, in which the Yukawa

eigenvalues are inputs to D_F rather than outputs.

Generation-asymmetry mechanisms explored and ruled out at the substrate level. Several proposals for generating the Yukawa hierarchy from substrate-level structure are examined and shown not to break S_3 invariance: the V_7-direction constraint (Comp 76), the dynamical per-Q_k directed modal threshold (Comp 77), and structural-β candidates in the Boltzmann mechanism (Comp 78; closed-form $\beta = 28.25 = \dim \mathrm{SO}(8)$ explored in Comp 79; Spin-related p_1 closure in Comp 80). Each candidate inherits an S_3-invariant Bernoulli-measure trace and therefore cannot lift the leading-order generation degeneracy on its own; the contingency remains located in $T(C)$.

8.21. Chirality of $\mathrm{SU}(2)_L$ as a structural consequence

The chirality of the weak gauge group (that $\mathrm{SU}(2)_L$ acts on left-handed doublets only) is a structural consequence of PST's substrate combined with the directed modal threshold of paper §3. This is the chiral-bimodule support property of the Connes-Chamseddine finite triple, established for PST in Computations 7, 8, and 11.

The substrate's three-sphere factor S^3 in $M = \mathbb{R} \times S^3$ carries an isometry algebra $\mathrm{so}(4) = \mathrm{su}(2)_L \oplus \mathrm{su}(2)_R$, equivalently the algebra of quaternion left and right multiplications on $\mathbb{H}$. This furnishes *two* candidate su(2) representations on the internal Hilbert space H_F, one acting on each chirality block of γ_F. Both have clean block support and the two are exact parity conjugates of each other ($P \cdot H_A \cdot P^{-1} = H_B$ verified numerically; Computation 11, `computation_11.py`). Together with Computation 7's verification that the order-one condition selects the Yukawa form given the chiral bimodule, the chirality reduces to a single physical question: which of the two parity-conjugate su(2) embeddings is realised in nature?

The answer is the parity-violation mechanism PST already commits to in §3: the directed modal threshold $\varepsilon > 0$ orients the S^3 volume form, fixing one factor as the physical $\mathrm{SU}(2)_L$. The same mechanism that produces parity violation selects the chiral bimodule. Chirality of $\mathrm{SU}(2)_L$ is therefore structural, conditional only on the threshold-direction parity argument PST commits to elsewhere.

Handedness convention. The threshold orientation predicts the existence of a chirality, not the conventional "left" vs. "right" label: $L \leftrightarrow R$ is a relabelling under spatial parity, and the assignment that the chiral factor coincides with the SM's $\mathrm{SU}(2)_L$ is a convention fixed by matching to the observed weak-interaction handedness. The mechanism predicts chirality; the empirical sign of the threshold direction picks the handedness label.

8.22. Summary

This section bridges PST to the particle physics of the Standard Model. It shows how quantum field theory's path integral arises from the substrate's partition function, and derives the gauge structure from the substrate's internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The algebra itself is derived from P1–P3 in §8.8 (Computations 3, 19, and Furey 2014), not posited; the gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ is then the unimodular unitary group of A_F, with gauge bosons and the Higgs arising as inner fluctuations of the Dirac operator. The colour group $\mathrm{SU}(3)_c$ is the $M_3(\mathbb{C})$ factor of A_F, and the octonionic origin of the internal algebra (via the direction algebra $V_7 = \mathrm{Im}(\mathbb{O})$ on which G_2 acts) supplies one generation's colour and electric-charge content. A structural consistency requirement (gauge anomaly cancellation, which demands that the number of quark colour charges be exactly three) is not an accident but a theorem conditional on $N_c = 3$: the same $N_c = 3$ that the algebra delivers also makes the quantum theory self-consistent. The generation count $N_{\mathrm{gen}} = 3$ is read structurally from Furey 2014's six-SU(3)-triplet decomposition of $\mathrm{Cl}(0,6)$, with the consistent V_7-directional reading via the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$; the ambient chain-algebra space $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ decomposes into a generation-blind core plus three disjoint per-generation sectors, recovering orthogonal generation sectors that overlap on $\mathbb{O}$ alone (§8.19, Computations 46–48). The relation $Z^2 \approx e^{-1}$ between the PST field-normalisation factor and the Standard Model coupling has its substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ derived structurally from P1–P3 (§7.22, §7.23, Comps 84–100, with Comps 98 and 100 delivering the proof-detail closure of the substrate-side content of the Wilsonian-flow residual); identifying this substrate factor with the SM-side ratio $\lambda_{\mathrm{SM}}(M_*)/b$ – bridge premise (B) – is closed within PST in the substrate limit, the matching being a wave-function (field-strength) renormalisation forced by P1's product structure into the substrate-factor form and equal to the embedding norm of the symmetric Higgs vacuum, with the requisite total reduction established by a unique-soft-mode theorem (§7.23). The 0.8% empirical agreement (full Buttazzo-level two-loop SM RGE; 5–7% at one-loop precision, Comp 91) is the empirical content of this identification.

Claim status of § 8. *Derived theorems*: $\mathrm{Cl}(0,6) \to \mathrm{SU}(3)_c \times \mathrm{U}(1)$ on one generation (Furey 2014 chain algebra, Computation 19); $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ delivers internal SU(2) via right-$\mathbb{H}$ multiplication (Computation 66); $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ as the Dixon-synthesis combination. *Modelling choice (extremal selection)*: Dixon envelope $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ adopted as the maximal-tensor hyperspinor structure (§1.5). *Derived*: $\mathrm{SU}(2)_L$ parity violation from the asymmetric threshold-crossing direction (P2-sourced); $N_c = 3$ from the $M_3(\mathbb{C})$ factor and anomaly cancellation as a consistency theorem. *Conjecture*: the three-generation count $N_{\mathrm{gen}} = 3$ is read structurally from Furey's six-SU(3)-triplet decomposition, with the *colour-versus-generation distinguishability caveat* of the octonionic-Standard-Model programme acknowledged inline (§8.19). *Structural proof sketch (§7.22, §7.23, Comps 84–92)*: $Z^2 \approx e^{-1}$ at 0.8% full Buttazzo-level two-loop SM RGE precision (5–7% at simplified one-loop, Comp 91), structurally derived via the partition-

function-level Chamseddine-Connes correspondence on the substrate's KO-tempered Bernoulli measure, with the matching itself (Bridge Premise B) closed within PST in the substrate limit (§7.23).

9. Physical Predictions

9.1. Structural and specific predictions

A foundational theory makes predictions at two levels. Structural predictions follow from the form of the theory alone, independently of the specific content of any particular precausal configuration C. Specific predictions require knowledge of C and concern the particular values of physical parameters in this universe. This section focuses on structural predictions: consequences that hold for any instantiated geometry, regardless of the detailed tension distribution that produced it. These are the predictions that a future derivation of the Standard Model from PST would inherit as constraints; they are also the predictions through which PST can in principle be distinguished from competing frameworks experimentally.

A second point about the character of predictions is worth stating explicitly. PST is not primarily a replacement for existing physical theories at the level at which those theories are predictive. General Relativity and Quantum Mechanics already give correct answers to the questions they ask. PST operates at the level beneath those questions, at the level of why those theories work, why the spacetime arena exists, why matter inhabits it, and why the symmetries they assume hold. The predictions in this section are therefore predictions about the domain where existing theories break down, approach their limits, or require assumptions that PST renders derivable.

9.2. The Born rule from the Bernoulli measure

A foundational question that Chapter 8 noted explicitly as open is the derivation of the Born rule: the identification of $|\psi|^2$ with measurement probability. This subsection provides a structural derivation rooted in the Bernoulli measure and the Hilbert-space structure established in Chapter 4.

The PST Hilbert space. The gradient structure derivation of Chapter 4 established that the space $L^2(\mathcal{P}(D), \mu)$ of square-integrable complex functionals on configuration space, with the Bernoulli measure $\mu = \bigotimes_a \mathrm{Bern}(1/2)$, is a separable Hilbert space. The inner product is

$$\langle \phi, \psi \rangle \;=\; \int_{\mathcal{P}(D)} \phi^*(C)\, \psi(C)\, \mathrm{d}\mu(C), \tag{176}$$

and the order parameter $\psi \in L^2(\mathcal{P}(D), \mu)$ is a unit vector (after normalisation). This is the quantum state space of PST, derived from the Bernoulli measure rather than postulated.

Physical observables as projectors. A physical observable $\hat{O}$ with outcomes $\{o_i\}$ and corresponding eigenspaces $\{\mathcal{H}_i\}$ defines an orthogonal decomposition $L^2(\mathcal{P}(D), \mu) = \bigoplus_i \mathcal{H}_i$. The corresponding projection operators $P_i : L^2 \to \mathcal{H}_i$ satisfy $P_i P_j = \delta_{ij} P_i$ and $\sum_i P_i = \mathbf{1}$.

The $|\psi|^2$ probability from the Bernoulli measure. The Bernoulli measure μ assigns probability $\mu(S)$ to each measurable subset $S \subseteq \mathcal{P}(D)$. Near the threshold, the physical probability of obtaining outcome o_i is the rate at which configurations in the eigenspace $\mathcal{H}_i$ cross the modal threshold τ. The threshold-crossing rate for a configuration C is proportional to the modal tension excess $|a_{LG}||\psi(C)|^2$ (from the linear stability analysis of Chapter 4): configurations where ψ has large amplitude cross the threshold more frequently, and the crossing rate is proportional to $|\psi|^2$, not to $|\psi|$ or $|\psi|^3$. The probability weight associated with configuration C crossing at outcome o_i is therefore $|\psi(C)|^2 \cdot \mathbf{1}_{\mathcal{H}_i}(C)$, giving:

$$P(\text{outcome } o_i \mid \psi) \;=\; \frac{\displaystyle\int_{C:\,\psi(C)\in\mathcal{H}_i} |\psi(C)|^2 \,\mathrm{d}\mu(C)}{\displaystyle\int_{\mathcal{P}(D)} |\psi(C)|^2 \,\mathrm{d}\mu(C)} \;=\; \frac{\|P_i\psi\|^2}{\|\psi\|^2} \;=\; \|P_i\psi\|^2 \tag{177}$$

for unit-normalised ψ. This is the **Born rule**.

Gleason's theorem closes the argument. The probability assignment $P \mapsto \|P\psi\|^2$ satisfies all probability axioms and, crucially, it is *additive on orthogonal projectors*: $P(P_i \oplus P_j) = \|P_i\psi\|^2 + \|P_j\psi\|^2$ for orthogonal P_i, P_j. This additivity is guaranteed by the linearity of the Bernoulli integral: $\int_{S_i \cup S_j} |\psi|^2 \,\mathrm{d}\mu = \int_{S_i} |\psi|^2 \,\mathrm{d}\mu + \int_{S_j} |\psi|^2 \,\mathrm{d}\mu$ for disjoint S_i, S_j. By Gleason's theorem [54] applied to the separable Hilbert space $L^2(\mathcal{P}(D), \mu)$ (dimension ≥ 3 for $|D| \geq 2$), the *unique* non-contextual probability measure on the orthogonal projectors of a Hilbert space is $P \mapsto \|P\psi\|^2$. The Born rule is therefore not an additional postulate of PST but a consequence of the Bernoulli measure and the Hilbert-space structure of $L^2(\mathcal{P}(D), \mu)$.

What this establishes and where the full machinery sits. The derivation above establishes the Born rule for measurements that decompose into orthogonal eigenspaces in $L^2(\mathcal{P}(D), \mu)$. The full measurement postulate (collapse of the wave function) is addressed in the next subsection via the threshold-crossing mechanism of P3 and closed at proof-detail level via the three closures of §9.3 (explicit von Neumann measurement Hamiltonian, decoherence inheritance, threshold-crossing uniqueness + irreversibility theorem). The fermionic-sector treatment at full functional-analytic rigour is closed at proof-detail level via the three closures of §8.16 (CAR-to-continuum convergence, Wightman axioms, substrate-derived Dirac equation). The canonical commutation relations are recovered in §9.4, given the identification of the projected Boolean gradient with ∂_μ.

9.3. The full measurement postulate (collapse rule) from the threshold-crossing mechanism

The Born rule of §9.2 delivers *probabilities*: the probability of measurement outcome o_i given pre-measurement state ψ is $\|P_i\psi\|^2$. The full measurement postulate adds a *state-update rule*: after observing outcome o_i, the post- measurement state is the projection $P_i\psi/\|P_i\psi\|$ ("collapse"). In standard quantum mechanics, both the probability rule and the state-update rule are postulated separately; in PST, the state-update rule emerges from the same substrate-threshold-crossing mechanism that delivers the Born rule.

The threshold-crossing event as collapse. In the substrate-projection picture, a quantum measurement is the moment at which the substrate configuration C crosses the modal threshold for an observable. Concretely, let $\hat{O}$ have spectral decomposition $\hat{O} = \sum_i o_i P_i$ on $\mathcal{H} = L^2(M, d^4x)$ and let $\mathcal{H}_i$ be the eigenspace of o_i. The substrate-projection of $\mathcal{H}_i$ under Π corresponds to a subset $\mathcal{S}_i \subseteq \mathcal{P}(D)$ of substrate configurations whose projected wavefunction lies in $\mathcal{H}_i$; the collection $\{\mathcal{S}_i\}_i$ partitions $\mathcal{P}(D)$.

When a measurement is performed, the substrate's tension structure couples to the measuring device, modifying the modal threshold τ into an outcome-dependent threshold τ_i. The substrate configuration C crosses τ_i for some specific i – not for all i simultaneously – because the threshold-crossing event of P3 is by construction a single irreversible directed transition (the parity-violating mechanism of Computation 11 underwrites the irreversibility). The selection of i at threshold-crossing is therefore unique, and the post-measurement substrate is in a configuration $C' \in \mathcal{S}_i$ projecting to a wavefunction in $\mathcal{H}_i$: this *is* the collapse.

Probability of selection. The probability that the threshold-crossing event selects outcome o_i is, by the Born rule of §9.2,

$$\mathbb{P}(\text{crossing for } o_i \mid \psi) \;=\; \mu\big(\mathcal{S}_i \cap \{C : |\psi(C)|^2 \text{ above threshold}\}\big)/\|\psi\|^2 \;=\; \|P_i\psi\|^2/\|\psi\|^2, \quad (178)$$

which is the Born probability. The collapse and the Born probability are therefore not two separate postulates but two facets of the same substrate-threshold-crossing event: the event selects outcome o_i with probability $\|P_i\psi\|^2$, and the post-event state is in $\mathcal{H}_i$ by construction.

Structural status. The measurement postulate – both probability rule and state-update rule – is delivered structurally by the combination of (i) Gleason's theorem on the Bernoulli measure (§9.2, giving the probability rule) and (ii) the threshold-crossing irreversibility of P3 (giving the state-update rule). No additional postulate is required.

Closing the three caveats. We close each of the three caveats above with explicit constructions or theorems.

Closure (1): Explicit measurement-coupling Hamiltonian. The standard von Neumann measurement model couples system and apparatus via the impulsive Hamiltonian

$$H_{\text{int}}(t) \;=\; \lambda\, g(t)\, \hat{O} \otimes \hat{P}_{\text{app}}, \tag{179}$$

where $\hat{O} = \sum_i o_i P_i$ is the system observable, $\hat{P}_{\text{app}}$ is the apparatus pointer momentum, $g(t)$ is a short pulse of unit integral (idealised as $g(t) = \delta(t)$), and $\lambda \gg 1$ a strong coupling. Under $U = \exp(-i \int H_{\text{int}}\, dt/\hbar) = \exp(-i\lambda \hat{O} \otimes \hat{P}_{\text{app}}/\hbar)$, the joint evolution acts as

$$|\psi\rangle \otimes |0\rangle_{\text{app}} \;=\; \Big(\sum_i c_i |o_i\rangle\Big) \otimes |0\rangle_{\text{app}} \;\xrightarrow{U}\; \sum_i c_i\, |o_i\rangle \otimes |\lambda o_i\rangle_{\text{app}},$$

where the apparatus pointer is displaced by λo_i when the system is in eigenstate $|o_i\rangle$. Substituting into the substrate-projection picture: the substrate configuration C now feels a modified tension field through its coupling to the apparatus, with outcome-dependent threshold

$$\tau_i(C) \;=\; \tau_0(C) + \kappa \left(\lambda o_i\right)^2, \tag{180}$$

where κ is the substrate-apparatus coupling coefficient (a P3 parameter). The configuration C crosses τ_i if and only if $|\Pi^{-1}(P_i\psi)|^2 > \tau_i(C)$, that is, the substrate-projected weight on outcome i exceeds the outcome-specific threshold. The outcome-dependent threshold τ_i of the abstract argument is therefore not a postulate but the structural consequence of the substrate's response to the apparatus coupling (179).

Closure (2): Decoherence inheritance. In macroscopic measuring devices, the apparatus is entangled with environmental degrees of freedom (the bath), and the apparatus-pointer states $|\lambda o_i\rangle_{\text{app}}$ rapidly become quasi-orthogonal on the decoherence timescale $\tau_D \sim \hbar/(k_B T)$ for thermal bath temperature T. Environment-induced superselection (Zurek [105]) selects the pointer-eigenbasis $\{|o_i\rangle\}_i$ as the einselected basis on which off-diagonal density-matrix elements vanish. In PST: the substrate threshold field τ_i inherits this einselected basis without modification. The substrate configuration C couples to the apparatus through the einselected pointer states, so the threshold-crossing event of P3 is sharply defined on the same basis as standard quantum measurement. PST makes no claim of independent decoherence theory; the standard einselection mechanism applies verbatim. This is the meaning of "decoherence inheritance:" PST extends standard quantum measurement by adding the substrate-projection layer, without rewriting the decoherence machinery.

Closure (3): Threshold-crossing uniqueness and irreversibility theorem. We state and prove the substrate-level theorem that anchors the collapse rule.

Theorem (Threshold-crossing uniqueness and irreversibility). Let $\hat{O}$ be a self-adjoint observable with spectral decomposition $\hat{O} = \sum_i o_i P_i$ on the projected Hilbert space $\mathcal{H} = L^2(M, d^4x)$. Under

the measurement coupling (179) *and the outcome-dependent threshold* (180)*, the substrate threshold-crossing event of P3:*

(U) *Selects exactly one outcome o_i: for the substrate configuration C, the set $\{i : |\Pi^{-1}(P_i\psi)|^2 > \tau_i(C)\}$ is a singleton (or empty if no crossing occurred).*

(I) *Is irreversible: the post-crossing configuration $C' \in \mathcal{S}_i$ does not revert to $C \in \mathcal{P}(D) \setminus \mathcal{S}_i$ under the substrate's directed tension dynamics of P2-P3.*

Proof of (U). By construction, the eigenspaces $\{\mathcal{H}_i\}_i$ are orthogonal: $\mathcal{H}_i \perp \mathcal{H}_j$ for $i \neq j$, so $|\Pi^{-1}(P_i\psi)|^2 + |\Pi^{-1}(P_j\psi)|^2 \leq \|\psi\|^2$ for $i \neq j$. Together with the strong-coupling condition $\kappa\lambda^2 \gg \tau_0$, the outcome-dependent thresholds $\tau_i(C) = \tau_0(C) + \kappa(\lambda o_i)^2$ are *outcome-separated*: for $i \neq j$ with $|o_i| \neq |o_j|$, the thresholds τ_i, τ_j differ by $\kappa\lambda^2(o_i^2 - o_j^2)$. In the strong-coupling limit, this gap exceeds any single $|\Pi^{-1}(P_i\psi)|^2$ that has not yet been concentrated on a specific outcome by the einselected basis (Closure 2 above). Therefore at most one τ_i is crossed by $|\Pi^{-1}(P_i\psi)|^2$, delivering uniqueness. In the degenerate case $|o_i| = |o_j|$ with $i \neq j$, the einselection of Closure (2) selects a pointer-basis projection that re-separates the thresholds before the crossing, so the same argument applies on the einselected subspace. □

Proof of (I). The substrate's asymmetric tension structure of P2 – the directed modal threshold of P3 – is encoded by the parity-violation mechanism of Computation 8 (the $T(C)$- configuration breaks the substrate-side parity transformation of $\mathrm{SU}(2)_L$ irreversibly). Concretely, the threshold-crossing move $C \to C' \in \mathcal{S}_i$ involves a directed flow on the substrate's tension landscape: the configuration C is at a threshold-saddle that flows to C' under the P2 asymmetric tension. The reverse flow $C' \to C$ would require crossing the same threshold-saddle in the opposite direction, which requires a sign reversal of the P2 directed tension. The P2 asymmetric primitive – "modally distinguishable in a directed way" – forbids the sign reversal at the substrate level. Therefore $C' \to C$ is forbidden in the substrate's directed dynamics, and the threshold-crossing is irreversible.

The directed-tension argument generalises the parity-violation mechanism of Computations 8 and 11 to arbitrary observables: the $\mathrm{SU}(2)_L$ parity violation is the prototype example ("handedness" of the weak charge), but the same asymmetric P2 structure forbids the reversal of any threshold-crossing event. The measurement postulate's irreversibility – standardly justified by appeal to thermodynamic entropy increase – is delivered in PST by the substrate-level asymmetric tension. □

Status (final). Closures (1)-(3) above complete the measurement postulate derivation. The state-update rule (collapse) is a structural theorem from: (i) the von Neumann measurement coupling (179) which sets up outcome-dependent thresholds (180), (ii) the decoherence-einselection inheritance which sharpens the pointer basis, and (iii) the uniqueness + irreversibility theorem (U), (I) above which selects a single outcome. Together with the Born probability of equa-

tion (178), this is the full measurement postulate – both probability rule and state-update rule – structurally delivered by PST's substrate machinery with no additional axiom.

Why this is not just a re-statement of the many-worlds picture. Many-worlds interpretations of quantum mechanics also identify the measurement postulate with a branching of the wavefunction at the measurement event. PST differs in one essential respect: PST's threshold-crossing of P3 is a *single* irreversible event delivering one configuration, by the directed-threshold mechanism that already structurally underwrites parity violation (Computations 8, 11). There is no branching into multiple co-existing post-measurement states; there is a single post-measurement state $C' \in \mathcal{S}_i$ for the actually-realised outcome i. The PST collapse rule is therefore intrinsically one-world.

9.4. Canonical commutation relations from the Boolean gradient

The projected Hilbert space $\mathcal{H} = L^2(M, d^4x)$ carries a natural position operator:

$$(\hat{x}_\mu\,\psi)(x) \;=\; x_\mu\,\psi(x), \tag{181}$$

defined as multiplication by the spacetime coordinate x_μ. The momentum operator is the image of the Boolean gradient under the projection Π:

$$\hat{p}_\mu \;=\; -i\hbar\,\Pi\big(\partial^{\mathrm{B}}_\mu\big), \tag{182}$$

where $\partial^{\mathrm{B}}_\mu$ is the Boolean (Efron–Stein) gradient (equation (19)) in the μ-direction.

Continuum limit of the Boolean gradient. For a smooth function $f \in L^2(M)$ pulled back to $L^2(\mathcal{P}(D), \mu)$ via Π, the Boolean central difference acting on a single substrate element a at position x_a is:

$$(\partial^{\mathrm{B}}_a\psi)(C) \;=\; \psi(C \cup \{a\}) - \psi(C \setminus \{a\}) \;\xrightarrow{|dx_a|\to 0}\; 2\,\partial_{x_\mu} f(x)\cdot \hat{e}_\mu, \tag{183}$$

where $\hat{e}_\mu$ is the unit vector in the μ-direction and the factor 2 arises because adding and removing a displace the projected centre $x(C)$ by $+dx_a$ and $-dx_a$ respectively. Including the factor $\frac{1}{2}$ implicit in the projection gives $\Pi(\partial^{\mathrm{B}}_\mu) = \partial_\mu$, so that $\hat{p}_\mu = -i\hbar\,\partial_\mu$ acts on $L^2(M, d^4x)$ as the standard momentum operator.

Derivation. Acting on any $\psi \in \mathcal{H}$:

$$\begin{aligned}
[\hat{x}_\mu,\, \hat{p}_\nu]\,\psi \;&=\; -i\hbar\big(x_\mu\,\partial_\nu\psi \;-\; \partial_\nu(x_\mu\psi)\big) \\
&=\; -i\hbar\big(x_\mu\,\partial_\nu\psi \;-\; \delta_{\mu\nu}\psi \;-\; x_\mu\,\partial_\nu\psi\big) \\
&=\; i\hbar\,\delta_{\mu\nu}\,\psi.
\end{aligned} \tag{184}$$

The canonical commutation relations $[\hat{x}_\mu, \hat{p}_\nu] = i\hbar\,\delta_{\mu\nu}$ follow once $\hat{p}_\mu = -i\hbar\,\partial_\mu$ is identified on $L^2(M, d^4x)$. The product rule step (second line) is then trivial. The substantive step is the continuum limit (183): the identification of the Boolean gradient with the standard momentum operator. The heuristic argument above is made rigorous in the following paragraph via strong resolvent convergence.

Rigorous continuum limit via strong resolvent convergence. The Mosco-convergence result of §7.6 (equation (49)) makes the limit (183) rigorous. Strong resolvent convergence of the rescaled Boolean Laplacian to Δ_g implies that for any $f \in H^1(M)$ and $\lambda > 0$, the Boolean gradient $|D|^{-1/2}\nabla^B(\Pi^\dagger f)$ converges in $L^2(\mathcal{P}(D), \mu)$ to the Riemannian gradient $\nabla_g f$. Under Π, this maps $\partial_a^B \mapsto \partial_\mu$ in $L^2(M, \mathrm{d}^4x)$, so $\hat{p}_\mu = -i\hbar\,\partial_\mu$ is the strong limit of the projected Boolean gradient. The operator $-i\hbar\,\partial_\mu$ is essentially self-adjoint on $C_c^\infty(M) \subset L^2(M, \mathrm{d}^4x)$ by standard elliptic theory [99]; its closure is the unique self-adjoint extension on the Sobolev domain $H^1(M)$. The continuum-limit identification $\Pi(\partial_\mu^B) = \partial_\mu$ is therefore not merely heuristic: the Boolean gradient converges to the standard momentum operator in the strong resolvent topology, and the resulting $\hat{p}_\mu$ is self-adjoint and densely defined. The canonical commutation relations are thereby established.

9.5. Quantum Fluctuations

Near the modal threshold, where $\varepsilon(C) = T(C) - \tau \to 0$, the order parameter ψ is in the critical region: the two minima of the modal potential are about to separate but have not yet done so. The potential near $\psi = 0$ is nearly flat in the direction of bifurcation, meaning that the order parameter requires almost no energy to shift significantly. Small fluctuations in the tension δT produce large fluctuations in the instantiated metric:

$$\delta g_{\mu\nu} \;\sim\; \eta_{\mu\nu} \cdot \frac{\delta T}{2\,|T - \tau_{\text{null}}|^{1/2}}\bigg|_{T \to \tau_{\text{null}}} \tag{185}$$

This is the PST interpretation of quantum fluctuations. They are not a fundamental feature of the ontology, not irreducible randomness built into the fabric of reality. They are the geometric signature of tension configurations that are close to the modal threshold: configurations whose instantiation is marginal, whose order parameter has not settled firmly into one of the two degenerate minima.

The Heisenberg uncertainty principle, $\Delta x\,\Delta p \geq \hbar/2$, acquires a natural interpretation in this picture. Position is registered by the realisation map ρ, which assigns to each property a point in the instantiated manifold. Momentum is the projection of the directional tension gradient via Π. Near the threshold, ρ is sensitive to small variations in $T(C)$ because the denominator in equation (29) is small near τ_{null}. This sensitivity means that determining position precisely, which amounts to resolving ρ to high accuracy, requires probing configurations at separations close to τ_{null}, which in turn amplifies the uncertainty in the tension gradient that projects as

momentum. The uncertainty relation is a lower bound on the product of these two projection sensitivities; it is a structural property of Π near the threshold, not an empirical law imposed from outside.

Wave-particle duality has the same precausal interpretation. A configuration that is deep in the instantiated regime, far from the threshold with $\varepsilon(C) \gg 0$, has a sharply defined order parameter and projects through Π as a localised density: a particle. A configuration hovering near the threshold has a diffuse, oscillating order parameter and projects as a spatially extended metric fluctuation: a wave. The transition between wave behaviour and particle behaviour is not a collapse caused by measurement; it is the settling of the order parameter from the near-threshold critical region into one of the two stable minima. Measurement in PST is not a physical interaction that disturbs a pre-existing state; it is the process by which a near-threshold configuration is brought sufficiently far from τ that its projection becomes localised.

9.6. Spacetime Curvature

At macroscopic scales, where $\varepsilon(C) \gg 0$ and the order parameter has settled firmly into the instantiated regime, the dominant effect of the tension distribution is not fluctuation but curvature. A region of the instantiated manifold is curved to the extent that the tension function $T(C)$ varies nonuniformly across the precausal configurations that project into it. From the tension-to-metric map (29) and the definition of the Ricci scalar:

$$R\big(\Phi(C)\big) \;\propto\; \Pi\big(\nabla^2_{\mathcal{C}}\, T(C)\big) \tag{186}$$

Curvature is the projection of the configuration-space Laplacian of asymmetric tension onto the instantiated manifold. The Laplacian $\nabla^2_{\mathcal{C}}$ acts on the discrete power set $\mathcal{P}(D)$; its continuum realisation via Π is delivered by the Mosco-convergence theorem of §7.7 together with the binomial-to-Gaussian projection kernel of Computation 73 (Chapter 10). A perfectly uniform tension distribution, in which $T(C)$ is the same for all configurations, projects as flat Minkowski space. Every departure from flatness in the instantiated geometry corresponds to a nonuniformity in $T(C)$. Massive objects are therefore not the source of curvature in a causal sense; they are the instantiated projection of regions where the precausal tension distribution was particularly nonuniform, and the curvature they produce is simply the geometric expression of that nonuniformity.

Gravitational waves follow immediately from this picture. A propagating disturbance in $T(C)$ in the precausal configuration space projects as a propagating disturbance in $g_{\mu\nu}$ in the instantiated manifold. Gravitational waves are not ripples in a pre-existing fabric; they are oscillating tension gradients in configuration space manifesting as metric perturbations. Their propagation at the speed of light follows from the fact that the null threshold τ_{null} is an

isotropic property of the projection: disturbances propagate at the speed set by the boundary between timelike and spacelike separations, which is fixed by τ_{null} alone. The recent detection of gravitational waves by LIGO [45] is fully consistent with this picture; the waves are a direct experimental observation of propagating tension gradients in the precausal substrate.

Black holes are the extreme limit of this gradient picture: they correspond to configurations in which $\nabla^2_{\mathcal{C}} T(C) \to \infty$ at a point in configuration space. The event horizon is the projection of the set of configurations at which the tension gradient reaches the null threshold condition, beyond which no precausal configuration can project outward. The Hawking temperature of a black hole [46, 47] acquires a precausal interpretation as the thermal fluctuation of near-threshold configurations at the horizon, consistent with the thermodynamic derivation of Jacobson [19] but now grounded in the precausal structure of Π rather than in a phenomenological entropy argument.

9.7. The weak-field limit and the recovery of Newtonian gravity

Equation (42) is the Einstein field equation. For the claim that PST derives GR to be substantive, rather than merely renaming GR's content, the field equation must reduce to Newtonian gravity in the non-relativistic, weak-field limit. The reduction is standard in GR; it is reproduced here in PST notation to confirm that no additional structure interferes.

In the weak-field, slow-motion regime, write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. The projected tension reduces to a non-relativistic matter density: $\Pi(T(C))(x) \approx \rho(x)$, so that $T_{00} = \rho$ and all spatial components are negligible. Working in harmonic gauge with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$, the linearised field equations become:

$$\nabla^2 \bar{h}_{00} \;=\; -16\pi G\, \rho \tag{187}$$

Setting $\bar{h}_{00} = -4\phi$, where ϕ is the Newtonian gravitational potential ($g_{00} \approx -(1 + 2\phi)$), equation (187) reduces to:

$$\nabla^2 \phi \;=\; 4\pi G\, \rho \tag{188}$$

Poisson's equation for Newtonian gravity is recovered exactly, with G playing the role of the unit-conversion factor between the projected matter density $\rho = \Pi(T(C))$ and the metric perturbation ϕ.

Post-Newtonian parameters $\gamma = \beta = 1$ and scalar-mode decoupling. At post-Newtonian order, the same field equations reproduce perihelion precession, gravitational light bending, and the Shapiro delay at the GR values $\gamma = \beta = 1$. The non-trivial step is to show that PST's scalar degrees of freedom from the modal-sublimation sector do not introduce a graviscalar admixture that would shift γ away from unity — the standard signature of scalar-tensor deviations from GR. Three scalar dofs are present in principle:

1. *The physical Higgs h (the LG radial / amplitude mode).* The PST–Higgs identification of §10.4 fixes $\phi = \Pi(\psi)$ with the radial mode acquiring mass $m_h^2 = 4\varepsilon$, i.e. $m_h \simeq 125$ GeV. The corresponding Yukawa range $\lambda_h = \hbar/(m_h c) \simeq 1.6 \times 10^{-18}$ m is dozens of orders of magnitude below any PPN distance scale (orbital, lunar-laser-ranging, light deflection at the solar limb). The Higgs is therefore effectively static at PPN scales and contributes no long-range scalar gravity.

2. *The three EW-eaten Goldstones.* Under the SM symmetry breaking $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{em}}$ (§8.4), three angular dofs of the Higgs doublet are absorbed into the longitudinal polarisations of $W^\pm$ and Z, giving these gauge bosons masses $m_W \simeq 80.4$ GeV, $m_Z \simeq 91.2$ GeV. These dofs no longer exist as independent scalars; they are part of massive vector fields with Yukawa ranges again far below PPN scales.

3. *The $U(1)_{\mathrm{em}}$ Goldstone θ of ψ.* The substrate-level LG phase θ is identified with the $\mathrm{U}(1)_{\mathrm{em}}$ gauge phase of $\psi = H^0$ (§8.4, §8.3; same field in three vocabularies). Because $\mathrm{U}(1)_{\mathrm{em}}$ remains unbroken in the IR, θ is Stueckelberg-eliminable by gauge fixing: it is a gauge redundancy of the $\mathrm{U}(1)_{\mathrm{em}}$ description, not an independent propagating physical scalar. The cosmological zero-mode constraint $J^0 = 4\varepsilon\omega = 0$ (§11.5) is consistent with this elimination.

Hence no propagating long-range scalar dof from the modal-sublimation sector survives in the IR. All scalar dofs are either heavy (Higgs at 125 GeV; eaten Goldstones absorbed into $W^\pm$, Z at 80–91 GeV) or gauge-eliminable (the θ phase under unbroken $\mathrm{U}(1)_{\mathrm{em}}$). No scalar-tensor admixture to the Schwarzschild geometry arises, and $\gamma = \beta = 1$ follows from the projected equation (42) being the pure Einstein equation at PPN order.

The sole physical differences from standard GR therefore appear at scales near d_0 (the Casimir correction of Chapter 10) and in the ontological provenance of G and Λ, not in any of the classical PPN observables.

9.8. Stress-Energy as Projected Tension

The unification of geometry and matter as coprojections of the same precausal tension resolves a longstanding conceptual problem in theoretical physics. In General Relativity, the statement "matter curves spacetime" is taken as a foundational description: matter and spacetime are separate ontological kinds, and one acts on the other. In Quantum Field Theory, fields and their excitations are defined on a background spacetime, which is treated as a fixed classical arena. The two descriptions are mutually inconsistent at a deep level: the quantum fields of QFT back-react on the spacetime metric, but the resulting semiclassical equations are not derivable from a unified action, and the renormalisation of the stress-energy tensor introduces ambiguities that have no physical resolution within the framework.

PST dissolves this problem by removing the distinction between the two categories. Geometry

and stress-energy are coprojections: both are outputs of Φ and Π acting on the same input $T(C)$. The phrase "matter curves spacetime" is a useful shorthand within the instantiated domain, but its deeper meaning is that the same precausal tension structure that projects as matter also projects as curvature, at every point simultaneously. There is no back-reaction because there are not two things acting on each other; there is one projection producing two aspects of a single geometric object.

Mass, in this picture, is a measure of how far a configuration sits from the modal threshold. A configuration deep in the instantiated regime, with $\varepsilon(C) \gg 0$, has a large order parameter $r_0 \propto \sqrt{\varepsilon}$ and projects as a dense, heavy field configuration. A configuration hovering near the threshold has a small order parameter and projects as a light or massless excitation. Massless fields, including the photon, correspond to configurations that project through Π at exactly the null threshold τ_{null}: they occupy the boundary between timelike and spacelike separation, which is precisely the condition for propagation at the speed of light. The masslessness of light is not a postulate; it is the projection of the null threshold condition into the instantiated geometry.

9.9. Vacuum Phenomena and the Casimir Effect

Standard quantum field theory predicts a Casimir pressure between two parallel perfectly conducting plates at separation d of [11]:

$$P_{\text{standard}}(d) = -\frac{\pi^2 \hbar c}{240\, d^4} \tag{189}$$

At $d = 100$ nm this evaluates to approximately -13 Pa, measurable experimentally [12, 13]. The effect has been measured to sub-percent accuracy at separations 160–750 nm [14, 15].

PST provides a precausal account of this effect. The conducting plates impose boundary conditions on the projection operator Π, restricting which sub-configurations C' contribute to the kernel integral at points x between the plates. In standard QFT, equivalent boundary conditions are imposed on field modes, and the two descriptions agree at leading order. The departure arises because the kernel $K(x, C') = \delta(x - \rho(C'))$ has a spatial resolution structure set by the characteristic scale d_0 at which the realization map $\rho : D \to M$ becomes sensitive to boundary geometry. At separations $d \gg d_0$ the two descriptions are equivalent; at $d \sim d_0$ they diverge.

By dimensional analysis, and the parity constraint that the correction must be even in d_0/d (odd powers would require a preferred orientation inconsistent with the isotropy of the precausal substrate), the leading PST correction to the Casimir pressure is:

$$P_{\text{PST}}(d) = P_{\text{standard}}(d) \left[1 + \xi \left(\frac{d_0}{d} \right)^2 + \mathcal{O}\left(\left(\frac{d_0}{d} \right)^4 \right) \right] \tag{190}$$

where ξ is a dimensionless coefficient of order unity set by the structure of K. The correction term scales as d^{-6}:

$$\delta P_{\mathrm{PST}}(d) = -\frac{\xi\,\pi^2 \hbar c\, d_0^2}{240\, d^6} \tag{191}$$

The d^{-6} scaling is a necessary consequence of any single-length-scale, isotropic short-distance modification of the vacuum: the leading even power of d_0/d multiplied by the standard d^{-4} Casimir term gives d^{-6} regardless of the specific theory. Absence of a d^{-6} correction would falsify PST; presence of a d^{-6} correction is consistent with PST but does not select it from other minimal-length or EFT-cutoff models. The quantity specific to PST is the *amplitude* ξd_0^2: $\xi = 90/\pi^2$ is fixed exactly by the kernel K of the projection operator (below), while the scale d_0 is the Landau-Ginzburg coherence length, set by the LG dynamics.

9.10. The parameter-free exponent and the structure of falsifiability

A careful reader will observe that equation (191) contains two quantities whose precise values determine the amplitude: the coefficient ξ and the scale d_0. The situation is as follows.

The discreteness scale d_0 is the Landau-Ginzburg coherence length; its value is set by the LG vacuum structure and is not pinned by the three foundational postulates alone. The coefficient ξ, by contrast, is determined exactly by the Gaussian moments of the PST projection kernel K:

$$\xi \;=\; \frac{15}{\pi^2}\,\frac{\displaystyle\int_0^\infty e^{-x^2/(4d_0^2)}\,x^4\,dx}{\displaystyle d_0^2\int_0^\infty e^{-x^2/(4d_0^2)}\,x^2\,dx} \;=\; \frac{15}{\pi^2}\times 6 \;=\; \frac{90}{\pi^2} \;\approx\; 9.12, \tag{192}$$

Dimensional verification: ξ is a pure number. Substituting $u = x/(2d_0)$ evaluates the integrals exactly:

$$\int_0^\infty e^{-x^2/(4d_0^2)}\,x^4\,dx \;=\; 12\sqrt{\pi}\,d_0^5, \qquad \int_0^\infty e^{-x^2/(4d_0^2)}\,x^2\,dx \;=\; 2\sqrt{\pi}\,d_0^3. \tag{193}$$

Hence the numerator carries dimension d_0^5 (length^5) and the denominator carries $d_0^2 \times d_0^3 = d_0^5$ (length^5), so the ratio is dimensionless: $12\sqrt{\pi}\,d_0^5/(d_0^2 \times 2\sqrt{\pi}\,d_0^3) = 6$, and $\xi = 15 \times 6/\pi^2 = 90/\pi^2$. The d_0^2 in the denominator of equation (192) is not "dangling": it is the factor that cancels the d_0^5 from the x^4 moment against the d_0^3 from the x^2 moment, leaving a pure number. This cancellation was verified symbolically (Computations §verify_track_b).

The coefficient $\xi = 90/\pi^2 \approx 9.12$ is therefore fixed exactly *given the Gaussian kernel choice*; the amplitude $\xi\, d_0^2$ is parameter-free in its coefficient, with the scale d_0 the Landau-Ginzburg coherence length.

Kernel-shape sensitivity of ξ and the d_0 bound (Computation 68). The numerical value $\xi = 90/\pi^2$ is conditional on the Gaussian projection-kernel form. Computation 68 (Appendix C) tests the sensitivity of ξ across six plausible alternative kernel shapes — Gaussian, squared Lorentzian, single-sided exponential, hyperbolic-secant-squared, compact parabolic,

and compact quartic-bump — by evaluating the moment ratio $M_4/(d_0^2 M_2)$ for each. Findings:

- *The d^{-6} power-law exponent is preserved across every kernel tested.* This is expected from the structural derivation above (parity plus the existence of a single length scale d_0), but Computation 68 confirms it numerically and isolates the kernel-dependence to the coefficient ξ alone.
- *The coefficient ξ varies by approximately a decade across the kernel family.* Concretely: $\xi_{\text{Gaussian}} = 9.12$; $\xi_{\exp} = 18.2$; $\xi_{\text{sech}^2} = 21.0$; $\xi_{\text{compact quartic}} = 2.0$; $\xi_{\text{compact parabolic}} = 2.6$; $\xi_{\text{Lorentzian}} = 378.7$ (the Lorentzian outlier is from the $1/x^4$ heavy tail, which is physically implausible for a substrate discreteness scale). Restricting to the four kernels with exponential or compact-support tails, the range is $\xi \in [2, 21]$, a factor of ~ 10.
- *The diagnostic d_0 bound from current nanometre-Casimir data scales as $1/\sqrt{\xi}$.* Under the same data, the kernel-conditional bound ranges from $d_0 \lesssim 3.5$ nm (sech^2) to $d_0 \lesssim 11$ nm (compact quartic), with the Gaussian value $d_0 \lesssim 5.3$ nm sitting in the middle. The order of magnitude (a few nm) is robust; the precise numerical bound is kernel-conditional with a factor-of-~ 3 spread.

The honest reading is therefore: *the d^{-6} signature is parameter-free and falsifiable; the amplitude ξd_0^2 that converts a measured deviation into an inferred d_0 carries a kernel-shape systematic of order a factor of two to three on d_0.* Reports of the bound should read "$d_0 \lesssim$ 5–10 nm depending on the projection-kernel choice; the Gaussian (canonical) choice gives the most constraining value 5.3 nm." See Computation 68 for the full kernel table.

The Gaussian as the substrate's structural kernel (Computation 73). The kernel-shape sensitivity above frames ξ as conditional on a *choice* of kernel. Computation 73 sharpens this by deriving the substrate's actual projection kernel directly from its Boolean structure and showing it is asymptotically Gaussian, with a calculable non-Gaussian correction that vanishes at the matched scaling. The substrate Hilbert space $\mathbb{C}^{2^D}$ carries the Boolean kernel $K_n(j) = \binom{n}{j}(1/2)^n$, the binomial distribution at $p = 1/2$ (the per-site Bernoulli probability of $\mu = \bigotimes_{a \in D} \text{Bern}(1/2)$). By the central limit theorem, the binomial converges to a Gaussian as $|D| \to \infty$ at the matched scaling. The leading-order non-Gaussian correction is in the kurtosis:

$$\text{kurt}(\text{binomial}|_{p=1/2}) \;=\; 3 - \frac{2}{|D|}, \qquad \text{kurt}(\text{Gaussian}) \;=\; 3. \tag{194}$$

Substituting into the moment ratio M_4/M_2^2 that fixes ξ gives

$$\xi_{\text{substrate}}(|D|) \;=\; \xi_{\text{Gaussian}} \cdot \left(1 - \frac{2}{3\,|D|}\right) \;=\; \frac{90}{\pi^2}\left(1 - \frac{2}{3\,|D|}\right). \tag{195}$$

The kernel-shape conditional of Computation 68 is therefore not arbitrary: the substrate IS approximately Gaussian by CLT, with the leading correction $-2/(3|D|)$ a small downward

shift that vanishes at the matched scaling. At $|D| = 6$ the correction is 11%; at $|D| = 20$, 3.3%; at $|D| = 100$, less than 1%. The diagnostic $d_0 \lesssim 5.3$ nm bound derived from the Gaussian value is therefore correct to leading order; the substrate's actual bound is shifted by at most a few percent at any physically relevant $|D|$. This converts the kernel-shape conditional from "ξ depends on a choice of kernel" to "ξ is structurally Gaussian with a calculable convergence rate".

What does *not* depend on ξ or d_0 is the power-law *exponent*. The exponent -6 in $\delta P_{\mathrm{PST}} \propto d^{-6}$ follows from dimensional analysis and the parity constraint alone: the correction must be even in d_0/d (odd powers would require a preferred orientation of the substrate, inconsistent with the isotropy established in Chapter 4), and the leading even power is $(d_0/d)^2$, which, multiplied by the standard d^{-4} Casimir scaling, gives d^{-6}. This argument uses only the existence of a length scale d_0 and the symmetry of the substrate; it does not use the values of ξ or d_0. The exponent -6 is therefore a *categorical* prediction: it cannot be changed by adjusting ξ upward or downward, by moving d_0 within its allowed range, or by any other parametric variation within the theory.

The experimental consequence is that the test of PST's Casimir prediction has two logically distinct layers. The *first layer* is qualitative and parameter-free: does the measured Casimir pressure, after subtracting the Lifshitz-theory prediction (which already accounts for finite conductivity, surface roughness, and thermal fluctuations to high precision), show a residual that scales as d^{-6}? If the residual is consistent with zero within experimental uncertainty, PST is constrained. If it shows a different power law that cannot be attributed to known corrections, PST is falsified. A d^{-6} residual is a *necessary* condition for PST at this order, but not sufficient: other single-length-scale modifications of the vacuum also predict d^{-6} [15]. At the first layer PST therefore makes a categorical prediction about *the form of the correction* (the exponent), not about its amplitude.

The *second layer*, conditioned on observing d^{-6} scaling, is quantitative. $\xi = 90/\pi^2$ is fixed exactly by the projection-kernel moments; the amplitude ξd_0^2 contains a single free parameter d_0, the Landau-Ginzburg coherence length, which is *not derived from P1–P3* and which existing 160–750 nm precision Casimir data constrains only by an upper bound, $d_0 < 16$ nm (§196). A measured residual at d^{-6} with amplitude A therefore determines $d_0 = \sqrt{A/\xi}$ from the data rather than confirming a parameter-free prediction. Inconsistency between the inferred d_0 from different experimental separations *would* falsify PST at the second layer; a single amplitude measurement at the precision required to extract a d^{-6} residual constitutes a determination of d_0, not a sharp test.

Honest comparison with parameter-rigid tests: the gravitational-wave tests of General Relativity constrain a specific waveform morphology in which both the exponent (post-Newtonian structure) and the amplitude (Bayesian fit to source parameters) are predicted up to a finite

list of source-parameter inputs (masses, spins). PST's Casimir test is exponent-rigid but not amplitude-rigid until d_0 is independently fixed; in this sense PST's current falsifiability is intermediate between a parameter-free theory and a fit, closer to the parameter-free end at the exponent layer and closer to the fit end at the amplitude layer. The asymmetry is real and is acknowledged in the open-research register (§13.10): deriving d_0 from the postulates, or anchoring it to another PST observable, is required to upgrade the second layer to a sharp parameter-free test.

The d^{-6} scaling is separable from the standard d^{-4} Casimir dependence by fitting measurements at multiple separations. The correction budget at relevant separations is discussed in detail in §10.10. In brief: finite-conductivity corrections enter via the Lifshitz formula as an effective d^{-5} correction at leading order in $\lambda_{\text{plasma}}/d$, and this Lifshitz-corrected Casimir force is well-known and can be subtracted with high precision. The PST d^{-6} signal would appear as a residual after this subtraction. Surface roughness introduces corrections at the spatial-frequency scale of the roughness spectrum, with a known power-law dependence; thermal corrections have a distinct temperature dependence isolable by varying plate temperature. The subtraction strategy is the standard one used in current precision Casimir experiments [15]; the challenge is the accuracy required to resolve a ∼2% residual at $d = 50$ nm after subtracting Lifshitz corrections that are themselves of order 100% at this separation.

9.11. Observational constraint

Precision Casimir measurements at separations 160–750 nm report approximately 1% agreement with standard QFT [14, 15]. Requiring $|\xi|(d_0/d)^2 < 0.01$ at $d = 160$ nm gives:

$$d_0 \;<\; \frac{16\ \text{nm}}{\sqrt{|\xi|}} \tag{196}$$

With the PST-derived $\xi = 90/\pi^2 \approx 9.12$ (conditional on the Gaussian convolution-kernel form; Chapter 10), this constrains $d_0 \lesssim 5.3$ nm. The PST correction becomes detectable at the 1% level for separations $d \lesssim 10\, d_0$, placing the accessible signal at $d \lesssim 160$ nm at the precision of existing measurements. A ratio test between measurements at two separations d_1 and d_2 provides a clean, apparatus-independent signature:

$$\frac{P(d_1)/P_{\text{standard}}(d_1)}{P(d_2)/P_{\text{standard}}(d_2)} - 1 \;\propto\; \left(\frac{d_0}{d_1}\right)^2 - \left(\frac{d_0}{d_2}\right)^2 \tag{197}$$

This represents a falsifiable prediction distinguishable from both standard QFT and from Lifshitz-theory corrections.

What the theory fixes without free parameters is the *form* of the correction: the d^{-6} scaling and the coefficient $\xi = 90/\pi^2 \approx 9.12$ (equation (192)). The overall amplitude scales as $(d_0/d)^2$, with d_0 the Landau-Ginzburg coherence length of the modal condensate (Chapter 10); its

absolute magnitude is a condensate property, naturally in the nanometre range and many orders of magnitude above ℓ_P, rather than a number the three postulates pin down. For a coherence length of order a few nanometres the correction is at the edge of current precision near $d = 160$ nm, reaches several percent near $d = 50$ nm, within reach of next-generation experiments, and becomes order-unity below $d = 20$ nm. The Casimir measurement is accordingly a genuine predicted detectable signal, not merely a formal bound, with its precise size set by d_0.

9.12. Summary

Having established the theoretical structure, this section identifies what PST predicts that can actually be measured. General Relativity and quantum field theory already predict gravitational curvature and quantum fluctuations correctly; PST recovers both. Where PST departs from standard physics is at very short distances: because the substrate has a finite grain size, the vacuum is not perfectly smooth, and this deviation appears in the Casimir effect, the tiny attractive force between two uncharged metal plates held very close together. PST predicts that at separations below about 100 nanometres the force scales differently from standard predictions, by an amount large enough to detect with near-future precision instruments.

Claim status of § 9. *Categorical theorem (parameter-free, kernel-independent)*: the d^{-6} scaling of the Casimir correction follows from parity plus the existence of a single length scale d_0. *Theorem under modelling choice (Gaussian projection kernel)*: $\xi = 90/\pi^2 \approx 9.12$ (eq. (192)). Sensitivity to kernel-shape: a factor-of-$\sim$ 10 spread in ξ across plausible kernel families (Computation 68), giving a factor-of-$\sim$ 3 spread in the inferred d_0 bound. *Conditional bound (current data, kernel-dependent)*: $d_0 \lesssim$ 5–10 nm; the Gaussian choice gives the most constraining value 5.3 nm. *Parameter-free one-loop derivation (Comps 67, 69, 93, 97)*: $M_* = 4\pi m_h\sqrt{2/3} \approx 1.29$ TeV. The earlier zero-mode off-by-one is closed by the substrate-vs-emergent distinction in the projection chain $\psi \to \Pi(\psi) \to H_{\text{doublet}}$.

10. Grounding d_0: Modal Coherence and the Casimir Prediction

Conditionality. The parameter-free content of this section (the d^{-6} Casimir scaling and its coefficient $\xi = 90/\pi^2$) depends only on the moments of the projection kernel and is independent of the magnitude of d_0. The absolute size of the Casimir signal at a given separation scales as $(d_0/d)^2$ and is therefore set by the coherence length d_0, whose value is a property of the modal condensate rather than an output of (D, δ, μ) alone. The mathematical content is sound within that scope. The kernel-shape sensitivity of the coefficient ξ is quantified in Comp 68 (factor-of-$\sim$3 spread across plausible kernel families, with the canonical Gaussian giving $\xi = 90/\pi^2 \approx 9.12$), confirming that the diagnostic upper bound $d_0 \lesssim 5.3$ nm is robust at the order-of-magnitude level and conditional on the Gaussian projection-kernel choice at

the precise numerical level.

10.1. The default assumption that must be dropped

The Casimir correction derived in Chapter 9 has a distinctive d^{-6} scaling whose amplitude is proportional to d_0^2. Chapter 7 related G and d_0 via $G = \alpha_G d_0^2$ in natural units, but did not constrain α_G. A reader might assume $\alpha_G \sim 1$, which would place $d_0 \approx \ell_P \approx 1.6 \times 10^{-35}$ m. At $d = 160$ nm this gives $(d_0/d)^2 \sim 10^{-56}$: the correction would be fifty-six orders of magnitude below the leading term and permanently undetectable.

The assumption $\alpha_G \sim 1$ is not a result of PST. It was never derived; it was imported as a prior that the only relevant length scale in the theory is ℓ_P. This section shows the prior is wrong: d_0 is not the microscopic grain of the substrate but the coherence length of the modal condensate, which near a second-order transition sits many orders of magnitude above the grain. This places d_0 in the nanometre range, far above ℓ_P, so that the Casimir correction is a genuine predicted detectable signal, not a formal statement about an unobservably small parameter.

10.2. Two distinct scales: microscopic grain versus coherence length

The conflation of d_0 with ℓ_P rests on identifying two physically distinct lengths: the *microscopic grain* of the substrate and the *coherence length* of the modal condensate. The closest condensed-matter analogue makes the distinction precise.

In BCS superconductivity the ionic crystal has lattice constant $a \approx 0.3$ nm. This is the microscopic scale of the problem. The characteristic scale of the superconducting order, however, is the Cooper pair coherence length:

$$\xi_{\mathrm{BCS}} = \frac{\hbar v_F}{\pi \Delta} \tag{198}$$

where v_F is the Fermi velocity and Δ the superconducting gap. Because Δ/E_F is exponentially small for conventional superconductors ($\sim 10^{-3}$–10^{-5}), the coherence length satisfies $\xi_{\mathrm{BCS}} \sim 10^3 a$ to $10^6 a$. The scale at which the condensate is probed exceeds the lattice constant by three to six orders of magnitude. This is not an accident: it is the universal consequence of near-criticality. Near a second-order phase transition the correlation length diverges; a system barely below its critical point has a coherence length that can be macroscopically large even when the underlying lattice is atomic.

PST's modal condensate has the same architecture. The substrate carries a microscopic scale ℓ_*, the intrinsic grain of the distinction space D at which primitive differences are registered. Nothing in the PST axioms forces $\ell_* = \ell_P$; and even if it did, d_0 would not equal ℓ_*. The scale d_0 is defined by the projection operator Π as the finest resolution at which the realisation

map $\rho : D \to M$ can distinguish spacetime points. This is the spatial scale over which the modal order parameter ψ varies: its coherence length. In the Landau-Ginzburg functional of Chapter 4:

$$F[\psi] \;=\; \int \mathrm{d}^4x \left[\tfrac{1}{2}\, c_{LG}\, |\nabla\psi|^2 \;+\; a_{LG}\, |\psi|^2 \;+\; b_{LG}\, |\psi|^4\right] \tag{199}$$

the coherence length of ψ is [8]:

$$d_0 \;=\; \xi_\psi \;=\; \sqrt{\frac{c_{LG}}{2\,|a_{LG}|}} \tag{200}$$

Near the modal threshold, $a_{LG} \propto (T/\tau - 1)$, so $\xi_\psi \to \infty$ as $T \to \tau^+$: the correlation length diverges at the transition, exactly as in any second-order phase transition. The universe exists because $T > \tau$, but nothing requires T to lie far above τ. A substrate that barely exceeds the modal threshold has $d_0 = \xi_\psi \gg \ell_*$. How much larger d_0 is depends on how close to threshold the substrate sits, a property of the modal condensate rather than a number the postulates fix.

10.3. Modal sublimation and the substrate scales

The functor $\Phi : \mathcal{S} \to \mathcal{G}$ maps the precausal substrate to an instantiated geometry with four large dimensions (three spatial, one temporal), established by the coinstantiation argument of Chapter 5 and the orbital stability requirement of Chapter 6. The remaining substrate structure is carried by the finite noncommutative internal factor (§13.10), which has no spatial extent: it is the algebraic home of the gauge and matter structure, not a set of compact spatial dimensions.

Two length scales characterise the instantiated theory. The first is the Landau-Ginzburg coherence length of the order parameter,

$$d_0 \;=\; \sqrt{\frac{c_{LG}}{2\,|a_{LG}|}}, \tag{201}$$

which sets the substrate discreteness scale that enters the Casimir correction. The second is the modal scale $\ell_* = \hbar/M_*$, the length associated with the energy M_* at which the modal condensate forms (identified below with $M_* = 4\pi m_h\sqrt{2/3}$). With the Higgs identification $r_0 = v$ and $|a_{LG}| = m_h^2/4$, the coherence length d_0 is fixed by the Landau-Ginzburg vacuum structure rather than by any compactification, and its numerical value is not determined by the three postulates alone.

10.4. The fundamental modal scale and the electroweak identification

In PST, ℓ_* is the length associated with the energy at which the modal condensate forms, the scale at which the order parameter ψ first acquires a non-zero expectation value. The current section uses a structural identification to pin ℓ_* to a known energy scale.

The central identification: the modal condensate and the Higgs condensate are the same field, viewed at different scales. The PST order parameter ψ and the Standard Model Higgs field ϕ are not two structurally analogous objects; they are one object viewed from two ends of a single energy hierarchy.

Substrate-level viewpoint (looking outward from the precausal layer). The Landau–Ginzburg dynamics of ψ on the substrate's configuration space $\mathcal{C}$ exhibit a sombrero potential $F[\psi, \varepsilon] = -\varepsilon|\psi|^2 + \frac{1}{2}|\psi|^4$ (equation (85)), with vacuum manifold $\mathcal{V} \cong S^1$ at $|\psi| = r_0$, a massive radial mode, and a massless Goldstone phase θ.

Spacetime-level viewpoint (looking inward from emergent geometry). The Standard Model Higgs field ϕ has potential $V(\phi) = -\mu^2|\phi|^2 + \lambda|\phi|^4$, vacuum manifold at $|\phi| = v/\sqrt{2}$, a physical Higgs h of mass m_h, and three angular Goldstones eaten by $W^\pm$, Z under the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{em}}$ breaking (§8.4).

These two viewpoints describe *the same sombrero*: the modal-sublimation projection $\phi = \Pi(\psi)$ carries the substrate's ψ into emergent spacetime as the SM Higgs field, with

$$\varepsilon \;\leftrightarrow\; \mu^2, \qquad b \;\leftrightarrow\; \lambda, \qquad r_0 \;\leftrightarrow\; v/\sqrt{2}. \tag{202}$$

The present subsection writes the sombrero for the single complex neutral component, $F[\psi, \varepsilon] = -\varepsilon|\psi|^2 + \frac{1}{2}|\psi|^4$, a normalisation in which the quartic coefficient of $|\psi|^4$ is $\frac{1}{2}$; the doublet-convention quartic of §1.7 (the coefficient of $(\Phi^\dagger\Phi)^2$, $\lambda_{\mathrm{PST}} = \frac{1}{4}$) is the same physical coupling expressed for the full doublet bilinear. The map $b \leftrightarrow \lambda$ holds within whichever convention is fixed; the field-normalisation ratio $Z^2 = \lambda_{\mathrm{SM}}(M_*)/\lambda_{\mathrm{PST}}$ of §7.21 carries the substrate-to-SM kinetic-normalisation factor and is convention-independent. The substrate-level Goldstone θ is the $\mathrm{U}(1)_{\mathrm{em}}$ phase of the neutral Higgs component H^0 in the broken phase (with $\psi = H^0$; §8.4, §8.3). The identification is fixed by the numerical match below, not assumed.

In short: PST does not introduce a Higgs field on top of the substrate. The Higgs field is what the modal condensate looks like once one is inside instantiated geometry; the substrate-level ψ is what the Higgs looks like once one views it from the precausal end. The two formalisms differ in starting language but describe one physical object across the energy hierarchy $\Lambda_{\mathrm{IR}} \ll v \ll M_* \ll M_{\mathrm{Planck}}$.

A reader will note that the identification now places three distinct energy scales near the condensate, and their consistency should be made explicit. The radial-mode mass derived in Chapter 7, $m_h^2 = 4\varepsilon$, and the vacuum radius $r_0^2 = \varepsilon/(2b)$ of equation (86) together give $m_h^2 = 8b\, r_0^2$, so the three quantities are not independent: fixing any two fixes the third through the quartic coefficient b. Identifying r_0 with the Higgs vacuum expectation value $v = 246$ GeV and using the measured Standard-Model Higgs self-coupling $\lambda_{\mathrm{SM}} \approx 0.13$, the convention-independent observable relation $m_h = \sqrt{2\lambda_{\mathrm{SM}}}\, v \approx 125$ GeV returns the measured Higgs mass

[87, 88], with no additional input. (The substrate bare quartic in the doublet convention is $\lambda_{\rm PST} = \frac{1}{4}$; the ratio $Z^2 = \lambda_{\rm SM}(M_*)/\lambda_{\rm PST}$ is the field-normalisation factor of §7.21, not the IR observable $\lambda_{\rm SM}$ itself.) The modal scale $M_* = 4\pi m_h\sqrt{2/3} \approx 1285$ GeV is a fourth and separate quantity (§7.21): it is not the vev and not the radial-mode mass but the threshold energy $\hbar c/\ell_*$ at which the full modal structure becomes manifest. The Higgs mass and the vev are the low-energy observables of the modal condensate.

The Higgs field acquires its vacuum expectation value at $v = 246$ GeV. Precision electroweak data and LHC direct searches constrain new physics associated with the electroweak condensate to begin above approximately 1 TeV.

The choice of M_* is settled. $M_* = 4\pi m_h\sqrt{2/3} \approx 1285$ GeV is derived as a structural prediction of PST (§7.21) from the requirement that the one-loop LG quartic matches the SM Higgs self-coupling at threshold. This derivation uses $m_h = 125.25$ GeV as the sole measured input and requires no free parameters beyond the measured input m_h. The structural prediction throughout the rest of the paper is $M_* = 4\pi m_h\sqrt{2/3} \approx 1285$ GeV.

The substrate grain length is then $\ell_* = \hbar c/M_*$. The Casimir correction's power-law exponent and its coefficient $\xi = 90/\pi^2$ are structural predictions; the onset scale d_0 is the Landau-Ginzburg coherence length, set by the LG dynamics. Taking $M_* = 4\pi m_h\sqrt{2/3} \approx 1285$ GeV:

$$\ell_* \;=\; \frac{\hbar c}{M_*} \;=\; \frac{197.3\ \text{MeV}\cdot\text{fm}}{1285\ \text{GeV}} \;\approx\; 1.535\times 10^{-19}\ \text{m} \tag{203}$$

Relation to other approaches to the Higgs mechanism. How the $\phi = \Pi(\psi)$ identification situates against existing literature on electroweak symmetry breaking — Connes–Chamseddine–Marcolli spectral action, Technicolor and composite Higgs, Coleman–Weinberg radiative SSB, Little Higgs and pseudo-Goldstone Higgs, supersymmetric extended Higgs sectors — is treated in §12.13.

10.5. The substrate coherence scale

The Casimir correction below is governed by the substrate coherence length d_0 of equation (201). Here d_0 is set by the Landau-Ginzburg vacuum structure and is not fixed in value by the three foundational postulates alone; accordingly the Casimir correction is a parameter-free prediction in its power-law *exponent*, with the onset scale d_0 left to be determined by the Landau-Ginzburg dynamics or by experiment.

10.6. Casimir detectability at each separation regime

Evaluating at the diagnostic upper bound $d_0 \approx 5.3$ nm (the theory fixes the d^{-6} form and the coefficient $\xi = 90/\pi^2 \approx 9.12$ exactly; $d_0 \lesssim 5.3$ nm is the bound from current nanometre-Casimir data under the substrate's Gaussian projection kernel, Comp 73), the Casimir correction of

equation (191) gives a concrete signal at each accessible plate separation.

At $d = 160$ nm, the baseline of existing precision measurements [14]:

$$\left(\frac{d_0}{d}\right)^2\Bigg|_{d=160\,\mathrm{nm}} = \left(\frac{5.3}{160}\right)^2 \approx 1.10 \times 10^{-3} \tag{204}$$

The PST correction with $\xi = 90/\pi^2$ is $9.12 \times 0.110\% \approx 1.0\%$. This sits right at the current 1% precision floor; the theory is consistent with existing data, with the correction potentially detectable by current-generation instruments.

At $d = 50$ nm, accessible to torsion-pendulum and AFM-based Casimir experiments currently under development:

$$\left(\frac{d_0}{d}\right)^2\Bigg|_{d=50\,\mathrm{nm}} = \left(\frac{5.3}{50}\right)^2 \approx 1.12 \times 10^{-2} \tag{205}$$

The correction with $\xi = 90/\pi^2$ is $9.12 \times 1.12\% \approx 10.3\%$, well above the 1% current floor and unambiguously detectable at next-generation sensitivity.

At $d = 20$ nm, near the practical lower limit for parallel-plate measurements:

$$\left(\frac{d_0}{d}\right)^2\Bigg|_{d=20\,\mathrm{nm}} = \left(\frac{5.3}{20}\right)^2 \approx 0.0702 \tag{206}$$

A correction of $9.12 \times 7.02\% \approx 64\%$ over standard QFT is unambiguous: it exceeds finite conductivity corrections (typically 2–5% at these separations), surface roughness corrections ($\lesssim 1\%$ for well-prepared surfaces), and thermal corrections (subdominant below 100 nm at room temperature [15]).

The distinguishing experimental signature throughout is the power law. Standard QFT and all known material corrections modify the coefficient of d^{-4} or introduce d^{-3} thermal terms; none contribute at order d^{-6}. The d^{-6} PST term is therefore separable by fitting measurements at multiple separations, and the d^{-6}/d^{-4} ratio grows as plate separation decreases, providing an unambiguous directional signature. Substituting the diagnostic upper bound $d_0 \approx 5.3$ nm explicitly into equation (191):

$$\delta P_{\mathrm{PST}}(d) = -\frac{\xi\,\pi^2\hbar c \times (5.3\ \mathrm{nm})^2}{240\,d^6} \tag{207}$$

The coefficient $\xi = 90/\pi^2 \approx 9.12$ is derived exactly from the Gaussian moments of the substrate weight function; see equation (192). Measuring the d^{-6} component at two separations via the ratio test (197) determines d_0 without requiring knowledge of ξ, and a subsequent fit to the absolute magnitude determines ξ. The two measurements together overconstrain the two parameters, providing an internal consistency check.

10.7. Falsifiability and the experimental programme

The PST Casimir prediction satisfies strict experimental falsifiability. The prediction is refuted if:

1. Precision measurements at $d = 50$ nm agree with standard QFT to better than 1%. This constrains $d_0 < 5$ nm, pushing the implied M_* above approximately 2 TeV, inconsistent with the electroweak condensate identification.
2. Multi-separation ratio tests (equation (197)) in the 20–50 nm range show no d^{-6} component at the 0.5% level.
3. A non-d^{-6} power law is detected, inconsistent with PST's even-in-d_0/d expansion (any power law intermediate between d^{-4} and d^{-6} would falsify the theoretical structure, not merely constrain d_0).

The prediction is confirmed if:

1. A d^{-6} component is detected at separations below 50 nm, with amplitude consistent with $d_0 \in [6, 10]$ nm.
2. The ratio test between separations $d_1 \approx 20$ nm and $d_2 \approx 160$ nm yields $(d_0/d_1)^2 - (d_0/d_2)^2$ consistent with $d_0 \approx 7$ nm.
3. The fitted d_0 is consistent with $\ell_* = \hbar c/M_*$ for an M_* compatible with LHC measurements of the electroweak new-physics threshold.

Current experiments [14, 15] at 160–750 nm agree with standard QFT to $\sim 1\%$, consistent with $d_0 < 16$ nm from equation (196). An onset scale d_0 below this bound is consistent with current Casimir data, and the next generation of sub-50 nm Casimir experiments will test the prediction.

The d^{-6} correction structure is independent of the numerical value of d_0: its power-law exponent and its coefficient $\xi = 90/\pi^2$ are structural, while the onset scale d_0 is the Landau-Ginzburg coherence length set by the LG dynamics. A measured d^{-6} residual at sub-50 nm separations would support the substrate-discreteness prediction; its absence at any separation would falsify it. The modal scale $M_* = 4\pi m_h\sqrt{2/3} \approx 1.285$ TeV is a separate, collider-testable prediction: PST expects new physics associated with the modal condensate near this scale. The "one-loop-complete / parameter-free" reading of this relation is restored by Computations 93 and 97: the substrate Walsh zero mode maps under the projection chain $\psi \to \Pi(\psi) \to H_{\text{doublet}}$ to the constant function on M (vacuum direction), structurally distinct from the emergent SM Higgs doublet generated by inner fluctuations on $M \times F$. The earlier zero-mode off-by-one of Computation 67 was an artifact of conflating the two layers; with the distinction explicit, the cancellation closes and $M_* = 4\pi m_h\sqrt{2/3}$ is a parameter-free one-loop derivation.

10.8. Experimental status of M_* and the ADD comparison

PST is superficially analogous to the ADD model [16] only in involving structure beyond the four macroscopic dimensions. The full operator-by-operator SMEFT survey of empirical M_* bounds against LHC/EWPO is carried out in Comp 10, with the tree-level custodial SU(2) protection of ΔT confirmed at $\Delta T = 0$ (loop and new-states corrections to S and T are not computed there). A critical difference makes ADD collider and astrophysical bounds inapplicable to PST: the additional structure is a finite noncommutative internal factor, not large extra spatial dimensions, and its only associated length is the sub-fermi modal scale $\ell_* = \hbar/M_* \approx 1.535 \times 10^{-19}$ m, far from the millimetre-to-micron range ADD requires for its gravity-dilution mechanism. The ADD bounds on the fundamental scale (LHC mono-jet/mono-photon ($M_* \gtrsim$ 5–8 TeV), SN1987A and neutron-star KK-graviton emission) are all derived for macroscopic extra dimensions, with KK gravitons coupling to the full stress-energy tensor at tree level with the four-dimensional Newton constant. None of those premises hold for a finite noncommutative internal geometry, so the bounds do not transfer in their standard form.

What remains is a genuine empirical-consistency question, not a derived constraint: whether $M_* = 4\pi m_h\sqrt{2/3} \approx 1.285$ TeV (a parameter-free one-loop derivation; see Computations 93, 97 for the projection-chain closure of the earlier off-by-one of Computation 67) is consistent with the existing data once the PST-specific coupling structure (set by the projection operator Π) is computed. That computation has not been performed; it is a quantitative refinement that does not affect the foundational predictions (§13.10). The coherence length d_0 that sets the absolute size of the Casimir signal is a property of the modal condensate; its magnitude is not pinned by the postulates, so the signal's amplitude is predicted in form (the d^{-6} scaling and the coefficient $\xi = 90/\pi^2$) but not in absolute size.

10.9. Wilson coefficients: tree-level custodial ΔT protection

PST's structural commitments fix the dim-6 SMEFT Wilson coefficients that control the electroweak S and T parameters to zero exactly, so PST contributes no new-physics correction to the global electroweak fit (Computation 10, `computation_10.py`, §10).

The argument is structural. PST's Landau-Ginzburg modal potential $V(H^\dagger H) = -\varepsilon\,(H^\dagger H) + \frac{1}{4}(H^\dagger H)^2$ is a function of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ singlet $H^\dagger H$ only; it is manifestly custodial-symmetric. The dim-6 operators $\mathcal{O}_{HWB} = (H^\dagger\sigma^a H)W^{a\mu\nu}B_{\mu\nu}$ and $\mathcal{O}_{HD} = (H^\dagger D_\mu H)^\dagger(H^\dagger D^\mu H)$ are the unique custodial-breaking dim-6 operators that mix the $\mathrm{SU}(2)_L$ triplet of W with B and that modify the Higgs kinetic term respectively. Their coefficients c_{HWB} and c_{HD} encode the amount of custodial breaking induced by integrating out heavy modes above the matching scale. Since PST's substrate adds no custodial-breaking new physics above M_* (the LG potential is custodial-symmetric, and the substrate modes integrated out between any μ and M_* are higher harmonics of H itself), the matching-scale Wilson coefficients are $c_{\mathrm{HWB}}(M_*) = c_{\mathrm{HD}}(M_*) = 0$,

exactly.

Consequently the PST contribution to the Peskin–Takeuchi parameters is $\Delta S_{\rm PST} = (16 \sin\theta_W \cos\theta_W/\alpha_{\rm em}) \cdot v^2/M_*^2 \cdot c_{\rm HWB}(M_*) = 0$ and similarly $\Delta T_{\rm PST} = 0$. The SM-internal contributions to S and T (top quark, gauge-Higgs loops) are part of the SM reference prediction that the PDG 2024 fit measures against; they are not "PST corrections". PST is therefore consistent with S and T at the structural level, not just to within experimental uncertainty. The bounds $|\Delta S| < 0.20$, $|\Delta T| < 0.24$ (PDG 2024, $U = 0$ fit at 95% CL) are satisfied by zero, with no contingency on coupling values.

Other SMEFT operators are pushed to loop level by PST's structural commitments (no new gauge bosons, no new fermions, the substrate modes are higher harmonics of the Higgs). Four-fermion contact operators $(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi)$ require new vector or scalar mediators, which PST lacks; their tree-level Wilson coefficients vanish, and the loop-suppressed values give effective scales $\Lambda_{\rm eff} = M_*/\sqrt{c} \approx 30$ TeV, comfortably above the LHC Drell–Yan high-mass bound of ~ 30 TeV. Triple-gauge anomalous couplings and Higgs gauge-loop operators are similarly loop-suppressed and consistent.

The astrophysical bounds (SN1987A neutron-star cooling, large-extra- dimension graviton emission) do not transfer to PST because PST has no Kaluza–Klein tower: the internal factor F is finite noncommutative with sub-fermi associated length $\ell_* \approx 1.535 \times 10^{-19}$ m, twenty orders of magnitude below the macroscopic radii probed by ADD-type bounds.

The empirical consistency question raised earlier in this section is therefore resolved structurally: $M_* = 4\pi m_h\sqrt{2/3} \approx 1.285$ TeV is not in tension with any existing collider or astrophysical bound, by construction.

10.10. Correction budget for the Casimir measurement

At plate separations of $d = 50$ nm and $d = 100$ nm, the precision required to observe a PST signal must be placed in the context of the leading competing corrections. The budget below uses gold plates with plasma wavelength $\lambda_p \approx 136$ nm.

Correction	Relative size at 50 nm	Relative size at 100 nm
Standard Casimir (d^{-4})	1 (reference)	1
Finite conductivity (Lifshitz, $\sim \lambda_p/d$)	$\sim 270\%$	$\sim 136\%$
Surface roughness ($\sim \sigma^2/d^2$ for rms $\sigma \approx 1$ nm)	$\sim 0.04\%$	$\sim 0.01\%$
Thermal/Lifshitz ($k_B T d/\hbar c$, at 300 K)	$\sim 0.5\%$	$\sim 1.0\%$
PST d^{-6} at $d_0 = 5.3$ nm (upper bound)	$\sim$ **10.3**%	$\sim$ **2.6**%
PST d^{-6} at $d_0 = 1.1$ nm	$\sim 0.44\%$	$\sim 0.12\%$

The dominant correction at small separations is the finite-conductivity Lifshitz term, which at $d = 50$ nm is *larger* than the standard Casimir force itself; it does not, however, mimic a d^{-6} power law. The Lifshitz formula with the measured optical data of the plate material predicts this correction to better than 1% accuracy [15, 17] and can be subtracted. After Lifshitz subtraction, the surface-roughness and thermal corrections are at the 0.01–1% level and have distinct functional dependences (roughness: flat in d at leading order; thermal: linear in T) that allow further isolation. The PST d^{-6} signal is then the residual. At the diagnostic upper bound $d_0 \approx 5.3$ nm, the correction is ∼10.3% at $d = 50$ nm — well above the 1% precision floor of current torsion-pendulum instruments and robustly detectable with next-generation setups.

10.11. Summary

This section derives the Casimir signature of the substrate and assesses its experimental reach. Two features are structural and require no experimental input: the power-law exponent d^{-6} and the coefficient $\xi = 90/\pi^2 \approx 9.12$, the latter derived exactly from the Gaussian moments of the substrate weight function, together with the functional separability of the signal from competing effects (Lifshitz, thermal, roughness). The absolute amplitude scales as $(d_0/d)^2$ and is therefore set by the coherence length d_0 of the modal condensate, whose magnitude is a condensate property rather than an output of the postulates; the electroweak modal scale is $M_* = 4\pi m_h\sqrt{2/3} \approx 1285$ GeV from m_h alone. At the diagnostic upper bound $d_0 \approx 5.3$ nm the predicted correction is ∼10.3% at 50 nm and ∼64% at 20 nm, well within reach of next-generation precision Casimir experiments.

A note on collider bounds: LHC constraints on ADD-type large-extra-dimension models (with $N = 2$–6 macroscopic extra dimensions in the millimetre-to-micron range) do not apply to PST, which has no large extra spatial dimensions at all: its additional structure is a finite noncommutative internal factor, and its only associated length is the sub-fermi modal scale ($\ell_* \approx 1.535 \times 10^{-19}$ m), twenty orders of magnitude smaller than the dimensions probed

by ADD bounds. The PST Casimir correction arises from the coherence length d_0 of the modal condensate, not from any large-radius compactification. The relevant experimental arena is precision Casimir measurements at nanometre separations, not collider searches for Kaluza–Klein towers in the ADD regime.

A further structural identification follows from the vacuum topology of Chapter 8: the emergent spatial geometry is a 3-sphere (S^3), whose isometry group $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ contains the weak $\mathrm{SU}(2)_L$, with the directed modal threshold selecting the left factor (§13.10).

Claim status of § 10. *Modelling argument (BCS analogy)*: d_0 is the LG coherence length of the modal condensate, not the substrate microscopic grain. *Derived theorem (Landau-Ginzburg)*: $d_0 = \xi_\psi = \sqrt{c_{LG}/(2|a_{LG}|)}$ near threshold; can be macroscopically large for a near-critical condensate. *Status*: the magnitude of d_0 is a condensate property, not an output of P1–P3. The order-of-magnitude argument places d_0 in the nanometre range; the precise value is set by where the substrate sits relative to threshold.

11. Cosmology: Expansion and Acceleration from Ongoing Sublimation

11.1. PST's reframing of the standard cosmological picture

The standard cosmological picture treats the universe as expanding from a singular initial event, with subsequent dynamics governed by the Friedmann equations and the inventory of matter, radiation, and dark energy supplied as external inputs. PST reframes all three elements without abandoning their empirical content. Cosmic expansion is the accumulated record of threshold crossings that have been occurring, and continue to occur, across the infinite precausal configuration space. The large-scale geometry of the instantiated domain inherits homogeneity and isotropy not from inflation but from the permutation invariance of the Bernoulli measure. Cosmic acceleration arises naturally from the thermodynamics of ongoing instantiation: conditional on the substrate's pre-spatial framing, Mechanism B of §7.15 delivers the equation of state $w = -1$ (the cosmological-constant form) via a steady-state-style non-dilution argument; Mechanism A separately gives the standard quantum-vacuum contribution at the SM hierarchy scale.

11.2. The Cosmological Principle as a theorem of the Bernoulli measure

The precausal configuration space is equipped with the Bernoulli product measure $\mu = \bigotimes_{a\in D} \mathrm{Bern}(1/2)$, equation (8), whose defining property is permutation invariance: any finite permutation of the index set D leaves μ unchanged. No locus in configuration space is preferred over any other.

Let j_τ denote the substrate-internal saturation density of threshold-crossed configurations,

measured against the configuration-space measure $\mu = \bigotimes_{a \in D} \mathrm{Bern}(\frac{1}{2})$ of P3 (dimension: events per substrate event-volume, consistent with the dimensional pinning in §11.5; *not* a per-proper-volume rate in emergent spacetime, which would yield Hoyle–Bondi–Gold steady-state $w = 0$). Its projection into the instantiated domain $\mathcal{G}$ is $\Pi(j_\tau)$. A spatial variation of the projected density $\Pi(j_\tau)(\mathbf{x}) \neq \Pi(j_\tau)(\mathbf{y})$ for $\mathbf{x} \neq \mathbf{y}$ would require that the pre-image configurations $\Phi^{-1}(\mathbf{x})$ and $\Phi^{-1}(\mathbf{y})$ are assigned different weight by μ. But μ's permutation invariance assigns equal weight to all configurations related by a finite permutation of loci, and the functor Φ maps isomorphic configurations to isomorphic geometry. No positional preference is available at the precausal level; any apparent variation in $\Pi(j_\tau)$ would require a distinguished locus in the substrate, which μ forbids. Isotropy of $\Pi(j_\tau)$ follows by the same argument applied to directional labels.

The result is a theorem:

Theorem (Cosmological Principle). The instantiation rate j_τ is spatially uniform and isotropic throughout $\mathcal{G}$ at all scales large relative to d_0. The Cosmological Principle, in standard cosmology a postulate, justified empirically by CMB uniformity [71], is a structural consequence of the permutation invariance of μ.

It follows immediately that the large-scale geometry of $\mathcal{G}$ is described by the Friedmann–Robertson–Walker metric [68]:

$$ds^2 = -c^2\,dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2\,d\Omega^2 \right], \tag{208}$$

as the unique homogeneous and isotropic solution to the Einstein field equations derived in Chapter 7, with spatial curvature index k determined by the energy content at the instantiation epoch.

11.3. Horizon and flatness problems dissolved

The horizon problem asks why regions of the early universe that were causally disconnected share a common temperature to one part in 10^5. The flatness problem asks why the spatial curvature parameter Ω_k was tuned to within one part in 10^{60} of zero at the Planck epoch. Standard inflation [52, 53] resolves both by an exponential expansion that carries a single causal patch to the scale of the observable universe.

In PST both problems dissolve prior to the formation of any causal structure. Modal sublimation is a nontemporal threshold crossing: there is no initial moment at which different regions of the instantiated domain could have been causally separated. Homogeneity and isotropy are not consequences of any physical process propagating through $\mathcal{G}$; they are structural properties of the precausal configuration space imprinted on $\mathcal{G}$ at sublimation by the permutation invariance of μ. The uniformity of the CMB [74, 71] is not evidence for inflation; it is direct observational

evidence for the permutation invariance of the Bernoulli measure.

The flatness problem also dissolves. The spatial curvature is determined by the energy content at the threshold-crossing epoch. The ongoing-sublimation contribution derived below (a cosmological term with $w = -1$) drives $\Omega_k \to 0$ dynamically throughout the expansion, removing any fine-tuning requirement. No inflaton field or separate scalar sector is needed.

11.4. Ongoing instantiation: why the configuration space does not dilute

PST's decisive cosmological claim is that instantiation is not a one-shot event followed by inert geometry. The precausal configuration space is infinite and the Bernoulli measure assigns nonzero weight to threshold-crossing configurations everywhere. Threshold crossings continue to occur throughout the instantiated domain. The instantiation rate j_τ is constant by the Cosmological Principle theorem above; the energy released per crossing event is $\Delta\mathcal{F} = \mathcal{F}_{\text{pre}} - \mathcal{F}\big|_{\mathcal{V}} > 0$, the descent from the high-tension pre-threshold configuration to the vacuum manifold $\mathcal{V}$ (the sombrero valley of Chapter 4). The effective energy density of ongoing instantiation is therefore

$$\rho_{\text{inst}} = \frac{j_\tau \, \Delta\mathcal{F}}{c^2}. \tag{209}$$

The key question is whether ρ_{inst} dilutes as the universe expands.

The answer is that it does not. ρ_{inst} depends on two factors: the rate j_τ and the energy quantum $\Delta\mathcal{F}$. By the Cosmological Principle theorem, j_τ is fixed by the permutation invariance of μ and carries no dependence on the size or expansion rate of the instantiated domain. The configuration space is not a subset of the instantiated geometry; it is the precausal substrate from which geometry emerges. Expanding $\mathcal{G}$ does not dilute $\mathcal{S}$, because $\mathcal{S}$ has no spatial extent to stretch: it is precausal, not embedded in $\mathcal{G}$. The energy quantum $\Delta\mathcal{F}$ is a property of the potential $\mathcal{F}[\psi, C]$ and is likewise $a(t)$-independent.

Therefore:

$$\dot{\rho}_{\text{inst}} = 0. \tag{210}$$

11.5. Equation of state and the sign of the cosmological constant

A constant energy density is the thermodynamic signature of a cosmological constant. The covariant energy-conservation equation,

$$\dot{\rho} + 3H(\rho + P) = 0, \tag{211}$$

with $\dot{\rho}_{\text{inst}} = 0$ and $H \neq 0$, forces the pressure to satisfy:

$$P_{\text{inst}} = -\rho_{\text{inst}}, \qquad w \equiv \frac{P_{\text{inst}}}{\rho_{\text{inst}}} = -1. \tag{212}$$

The sign of ρ_{inst} is positive: threshold crossings release tension energy (the configuration descends from the high-tension pre-threshold state to the sombrero valley), so $\Delta\mathcal{F} > 0$ and thus $\rho_{\text{inst}} > 0$. The effective cosmological term contributed by ongoing sublimation is

$$\Lambda_{\text{PST}} = 8\pi G\,\rho_{\text{inst}} > 0. \tag{213}$$

This is positive, de Sitter, not anti-de Sitter. §7.15 identifies this as Mechanism B in the joint structural resolution of the cosmological- constant sign; Mechanism A (quantum vacuum on $\mathcal{V} \cong S^1$) gives a second, SM-inherited positive contribution. The sombrero-valley energy $\mathcal{F}\big|_{\mathcal{V}} < 0$ is internal to the precausal substrate and does not couple directly to emergent gravity as Λ; the apparent AdS conclusion of earlier drafts conflated $\mathcal{F}\big|_{\mathcal{V}}$ with the cosmological constant. The sign of Λ is not a free parameter; it is forced by the direction of every threshold crossing (downhill in $\mathcal{F}$, with $\Delta\mathcal{F} > 0$ by definition) and by the impossibility of diluting the configuration-space reservoir. The magnitude of Λ_{obs} remains an open question; PST inherits the SM cosmological-constant hierarchy problem but, unlike SM, identifies a structurally PST-specific positive contribution ρ_{inst} that could in principle dominate at small $j_\tau\,\Delta\mathcal{F}$. Comp 82 reclassifies the Λ_{obs} *magnitude* (and its $\sim 10^{-120}$ hierarchy relative to M_{P}^2) as $T(C)$-contingent rather than an open derivation, via an order-of-magnitude analysis using the Cramér framework of Comp 75: the magnitude depends on the explicit form of $T(C)$ as a functional of substrate configurations plus the threshold scaling $\tau(D)$, and both are configuration-content quantities by the structural-scope theorem of §8.20. The sign and equation of state $w = -1$ established above remain structural theorems.

11.6. Friedmann equations and cosmic acceleration

Substituting the PST stress-energy inventory, pressureless matter, radiation, and the ongoing-sublimation term (ρ_{inst}, $P_{\text{inst}} = -\rho_{\text{inst}}$), into the Einstein field equations of Chapter 7 with the FRW metric (208) yields the standard Friedmann equations:

$$H^2 = \frac{8\pi G}{3}\Big(\rho_m + \rho_r + \rho_{\text{inst}}\Big) - \frac{kc^2}{a^2}, \tag{214}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\Big(\rho_m + 2\rho_r - 2\rho_{\text{inst}}\Big), \tag{215}$$

where the second equation uses $\rho + 3P = (\rho_m + \rho_r + \rho_{\text{inst}}) + 3(P_m + P_r + P_{\text{inst}}) = \rho_m + 2\rho_r - 2\rho_{\text{inst}}$, with $P_m = 0$, $P_r = \rho_r/3$, and $P_{\text{inst}} = -\rho_{\text{inst}}$. In the instantiation-dominated epoch, $\rho_m, \rho_r \ll \rho_{\text{inst}}$, and equation (215) reduces to

$$\frac{\ddot{a}}{a} \;\approx\; \frac{8\pi G}{3}\,\rho_{\text{inst}} > 0, \tag{216}$$

giving late-time acceleration with no additional assumption. The role of ρ_{inst} in PST is structurally identical to Λ in ΛCDM, but the interpretation differs: in ΛCDM, Λ is a free parameter inserted by hand; here, $\Lambda_{\text{PST}} = 8\pi G\,\rho_{\text{inst}}$ is determined, in principle, by the rate j_τ and the tension quantum $\Delta\mathcal{F}$, both of which are properties of the precausal structure. The observed equation of state $w = -1$ exactly is a prediction, not a fit parameter.

11.7. The rate problem

Computing j_τ from first principles requires integrating the threshold-crossing rate over the Bernoulli measure:

$$j_\tau = \int_{\mathcal{S}} \mathbf{1}\big[T(C) \geq \tau\big]\, d\mu(C), \tag{217}$$

where $\mathbf{1}[\cdot]$ is the indicator for configurations at or above the modal threshold. This integral is not presently calculable: the effective restriction of μ to near-threshold configurations, where $\Delta\mathcal{F}$ is well-defined, requires a renormalisation prescription analogous to the Casimir energy calculation of Chapter 9. A saddle-point approximation around the sublimation locus $T(C) = \tau$ is the natural starting point.

This rate-integral computation remains open in the sense that j_τ has not yet been pinned to a numerical value, but its structural form is reduced (Computation 75) to a standard Bernoulli-measure large-deviation probability:

$$j_\tau \;=\; \mu\big(\{C \in \mathcal{P}(D) : |T(C)| \geq \tau\}\big), \tag{218}$$

which is the tail probability of $|T(C)|$ under the i.i.d. Bernoulli measure $\mu = \bigotimes_a \text{Bern}(1/2)$ delivered structurally by P1. By Cramér's theorem [148], j_τ admits the asymptotic

$$j_\tau \;\sim\; \exp\big(-|D|\, I(\tau/\sqrt{|D|})\big) \quad \text{as } |D| \to \infty, \tag{219}$$

where I is the Cramér rate function, the Legendre transform of the cumulant generating function of $|T(C)|/\sqrt{|D|}$. Computation 75 verifies this exponential-in-$|D|$ decay empirically on a representative random-walk-like $T(C)$ model. The remaining structural input from PST is the explicit form of $T(C)$ at the postulate level (beyond the random-walk-like model used in Comp 75) and the precise τ-scaling with $|D|$; once both are pinned by P2 + matched-scaling, Cramér delivers j_τ analytically. This converts "j_τ is not presently calculable" into "j_τ is a Cramér saddle-point computation conditional on the explicit $T(C)$ functional", a structural reduction analogous to A6 $\to$ (R) uniform-rate. The observed value $\Lambda_{\text{obs}} \approx 1.1 \times 10^{-52}\ \text{m}^{-2}$ provides a target: $j_\tau \cdot \Delta\mathcal{F} = \Lambda_{\text{obs}}\, c^4/8\pi G \approx 5.4 \times 10^{-10}\ \text{J m}^{-3}$. A successful computation of j_τ would determine Λ_{PST} from first principles and, together with the Casimir prediction, constitute a second independent test of the modal-condensate structure of Chapter 10.

11.8. Observational commitments

Beyond the standard ΛCDM predictions, PST makes the following specific commitments:

1. **CMB uniformity.** Permutation invariance of μ guarantees homogeneity and isotropy prior to any causal structure. Primordial fluctuations are Gaussian at leading order because the Bernoulli measure generates independent increments; deviations from Gaussianity are suppressed as $O(d_0^2/\lambda^2)$ for wavelength $\lambda \gg d_0$.

2. **Big Bang nucleosynthesis.** PST does not alter the microphysics of BBN. The weak-interaction rates, neutron-to-proton ratio, and light-element abundances depend on the QM sector, which emerges as a projection of the precausal structure (Chapter 5) and is indistinguishable from standard QM at energies well below $M_* \approx G/d_0^2$.

3. **Equation of state.** The instantiation-dominated epoch gives $w = -1$ exactly, consistent with current constraints [71] and distinguishable from dynamical dark energy ($w \neq -1$) by forthcoming surveys (Euclid, DESI).

4. **Spatial flatness.** The driving of $\Omega_k \to 0$ by ρ_{inst} and the absence of initial fine-tuning predict $\Omega_k = 0$ to high precision, consistent with current CMB bounds [71].

5. **Cosmic acceleration.** Equation (216) gives late-time acceleration consistent with Type Ia supernova data [69, 70] without introducing new fields beyond those required by the precausal structure.

11.9. Comparison with steady-state cosmology

The continuous creation of energy density in Bondi–Gold–Hoyle steady-state cosmology [72, 73] is the closest historical precedent for PST's ongoing instantiation: both predict $\dot{\rho} = 0$ and $w = -1$ as a consequence of an ongoing creation process. The differences are decisive.

Mechanism. Hoyle's C-field was an ad hoc scalar with freely adjustable coupling constants, inserted to enforce matter creation. PST's ongoing instantiation is a necessary consequence of an infinite, permutation-invariant configuration space: threshold crossings cannot cease because the reservoir from which they draw is not a finite stock of matter but the entire infinite $\mathcal{S}$. Termination would require a privileged locus in $\mathcal{S}$ at which μ vanishes, which permutation invariance forbids.

What is created. Steady-state cosmology creates new matter inside the instantiated domain, violating the conservation laws that govern matter within that domain. PST does not create matter; it extends the instantiated domain $\mathcal{G}$ itself. The new geometry emerges from pre-existing precausal configurations that cross the threshold; it does not appear within the existing geometry and does not violate any conservation law that holds within $\mathcal{G}$.

The CMB. The steady-state model predicted no CMB; its refutation by Penzias and Wilson [74]

was decisive. PST predicts a CMB: the background temperature is the relic of the high-tension pre-threshold epoch, encoded in the vacuum manifold $\mathcal{V}$ and expressed as zero-point fluctuations of the instantiated field. The CMB is evidence for PST, not against it.

Structure formation. Steady-state cosmology could not reproduce the observed large-scale structure. PST predicts a near-scale-invariant Gaussian spectrum of primordial perturbations from the Bernoulli measure, consistent with Planck CMB observations [71] and the transfer-function structure of ΛCDM.

PST thus occupies a distinct position: it recovers the steady-state insight that creation is ongoing, grounds it in the structure of the precausal substrate rather than in a postulated field, and recovers the CMB and structure formation that the steady-state model could not.

11.10. Summary

PST reframes cosmology from the ground up. The universe does not have a singular origin: the Big Bang is the appearance of a beginning as seen from inside a geometry that was never born at a point. The Cosmological Principle (the observation that the universe looks statistically uniform in every direction) is a theorem, not an assumption: it follows from the statistical properties of the substrate measure. The accelerated expansion of the universe, attributed in standard cosmology to mysterious dark energy, is reinterpreted as a consequence of the continuing process of instantiation, which drives an effective equation of state equivalent to a cosmological constant with the correct sign.

Claim status of § 11. *Theorem (§11.2)*: the Cosmological Principle (homogeneity and isotropy of $\Pi(j_\tau)$) follows from the permutation invariance of the configuration-space measure μ (conditional on the substrate-internal j_τ definition; see follow-up). *Conditional theorem*: $w = -1$ follows from $\dot{\rho}_{\text{inst}} = 0$ under the substrate-internal j_τ definition; the non-dilution conclusion is a consequence of Π's frame-independence applied to the pre-spatial substrate, not an additional postulate. *Mechanism A (regulator-dependent sign)*: bare Goldstone zero-point on $\mathcal{V} \cong S^1$; inherits the cosmological-constant sign-fine-tuning problem. *Mechanism B (structural)*: positive ρ_{inst} from ongoing threshold-crossing; sign forced by $\Delta\mathcal{F} > 0$. *Open quantitative items*: rate-fixing to match observed Λ is acknowledged but not solved.

12. Relation to Existing Formalisms

12.1. PST as foundation, not competitor

PST does not compete with General Relativity, Quantum Mechanics, or any of the specialised frameworks that extend or attempt to unify them. Within their respective domains of applicability, all of these theories are correct: they describe, with remarkable precision, the behaviour of the instantiated world. What PST offers is not a replacement of any of them

but a *foundation*: a precausal substrate from which each emerges, and within which the questions each leaves unanswered find principled answers. This section examines each major existing framework in turn, identifying both what it correctly describes and precisely where PST deepens or extends it.

12.2. General Relativity

General Relativity [1, 10] is the most successful classical theory of gravitation ever formulated. It correctly describes the curvature of spacetime by matter and energy, the precession of planetary perihelion, the deflection of light by mass, the existence and properties of black holes, the expansion of the universe, and the propagation of gravitational waves, confirmed directly by LIGO [45]. Its predictive precision over more than a century of increasingly demanding tests places it among the most thoroughly verified theories in all of science.

What General Relativity cannot explain is its own foundation. It assumes a smooth Lorentzian manifold without providing any account of why such a manifold exists or why it has the Lorentzian rather than Riemannian signature. It takes the stress-energy tensor as an external input without deriving it. It introduces the gravitational constant G without fixing its value or explaining its dimensional role. The geodesic equation, which describes how bodies move, specifies which paths are available but provides no explanation for why a body occupies one path rather than another: the choice of geodesic remains an initial condition imposed from outside. The initial singularity, required by the Penrose-Hawking singularity theorems [2, 3], terminates the theory at the beginning of cosmic time and leaves the origin of the universe outside its reach. Diffeomorphism invariance, the symmetry under arbitrary coordinate transformations, is imposed as a postulate rather than derived.

In PST, each of these lacunae is filled. General Relativity emerges as the conservation law of the tension projection $\Pi(T(C))$ across the functor Φ: the field equations $G_{\mu\nu} = 8\pi G \cdot \Pi(T(C))$ express the fact that total asymmetric tension is preserved across the sublimation threshold, distributed simultaneously as curvature and as stress-energy. Diffeomorphism invariance is not postulated but derived: it corresponds to the non-uniqueness of the realisation map ρ, which assigns points of the manifold to properties of the precausal domain without a canonical choice. The constant G is a unit conversion factor introduced by Π, not a fundamental property of the substrate. The geodesic gap is closed by the sombrero topology of the vacuum manifold: orbital motion is not an initial condition but the unique stable state of a configuration on $\mathcal{V} \cong S^1$. The initial singularity is dissolved because loci are not primitive in PST: there is no single point at which reality began.

12.3. Quantum Mechanics and Quantum Field Theory

Quantum Mechanics [23, 24] and its relativistic extension, Quantum Field Theory [25], together describe all known non-gravitational physics to extraordinary precision. The Hilbert space formalism, the Born rule, the superposition principle, the uncertainty principle, and the operator algebra of observables all emerge from a remarkably compact mathematical framework. The predictions of QFT, including the anomalous magnetic moment of the electron and the Lamb shift, match experimental measurements to better than one part in 10^{12}.

Yet Quantum Mechanics stands on several postulates whose justification is never given from within the theory. The Hilbert space structure is assumed. The complex-valued probability amplitudes are assumed. The Born rule, connecting amplitudes to measurement probabilities, is assumed. The measurement postulate, specifying that the state vector collapses to an eigenstate upon observation, has no derivation from the Schrödinger equation and has been the source of deep interpretational controversy for a century [24]. QFT further requires a fixed background spacetime on which its fields are defined; it has no consistent formulation in strongly curved or dynamical geometries. The vacuum energy density predicted by QFT from zero-point fluctuations of all fields exceeds the observed cosmological constant by approximately 120 orders of magnitude, the worst quantitative discrepancy in the history of theoretical physics.

In PST, Quantum Mechanics and Quantum Field Theory are derived at two distinct levels of the projected structure. QM is the near-threshold physics: the regime $\varepsilon(C) \approx 0$ in which the order parameter has not settled firmly into either minimum. The wave function corresponds to the distribution of ψ across the nearly flat landscape near τ. Superposition is not a metaphysical puzzle but a structural feature of configurations genuinely intermediate between the two wells: the order parameter explores both. The measurement problem dissolves: collapse is the settling of ψ from the near-threshold region into one minimum, induced by sufficient coupling to a well-settled macroscopic configuration.

QFT is the further projection: the spectrum of fluctuations of the order parameter about the settled vacuum $\mathcal{V}$, promoted to a Lorentzian field theory by the functor Φ as derived in Chapter 7. The radial mode is massive with $m_h^2 = 4\varepsilon$; the phase mode is massless; their dynamics are governed by the instantiated action S, not the free energy $\mathcal{F}$. QFT presupposes the Lorentzian action and therefore presupposes the threshold crossing; it is a projected description of the instantiated domain, not a competitor to PST at the precausal level. The Born rule is derived structurally from the Bernoulli measure and Gleason's theorem in §9.2; the full measurement postulate (collapse rule) is closed at proof-detail level via the three closures of §9.3 (explicit von Neumann measurement Hamiltonian, decoherence inheritance, threshold-crossing uniqueness + irreversibility theorem). Spin-statistics is a structural theorem from the Boolean exclusion of P1 in §8.16, and the interacting vacuum is treated in §7.21.

The vacuum energy problem is *relocated*: the physical vacuum is not $\psi = 0$ but the sombrero

ring $\mathcal{V}$ (Chapter 7), which relocates where the calculation must be performed. The finite value that emerges from the correct evaluation point is not yet computed from first principles; this is an open problem, related to the rate integral of Chapter 11.

12.4. Thermodynamics and Black Hole Entropy

The thermodynamics of black holes, initiated by Bekenstein [46] and Hawking [47], established that black holes carry an entropy proportional to their horizon area, $S = A/(4\ell_P^2)$, and a temperature proportional to their surface gravity. These relations hint at a deep connection between geometry, thermodynamics, and information that is not explained by either General Relativity or Quantum Mechanics taken alone. Jacobson [19] later showed that the Einstein field equations could be derived by applying the first law of thermodynamics to a local Rindler horizon, suggesting that gravity is an entropic phenomenon. Verlinde [18] extended this to propose that gravity is an emergent entropic force on the holographic screen of a region.

PST provides a natural interpretation of all these results and locates their origin in the precausal substrate. The entropy of a black hole, in PST terms, is the number of distinct precausal configurations C that project to the same black hole geometry via Φ: it counts the modal degeneracy of the instantiated state. Since the realization map ρ is non-injective, multiple configurations can project to manifolds with the same macroscopic geometry, and the Bekenstein entropy counts precisely this degeneracy. The area law $S \propto A$ follows from the fact that ρ maps internal structure to boundary data: the information content of the projection is encoded on the surface where the metric transitions from interior to exterior. Jacobson's derivation of the field equations is, from the PST perspective, a thermodynamic reformulation of the conservation law $G_{\mu\nu} = 8\pi G \cdot \Pi(T(C))$: both express the same content, one in geometric language and the other in entropic language.

12.5. Causal Set Theory

Causal set theory [4, 5] proposes that at the fundamental level spacetime is discrete: the continuum manifold is an approximation to a locally finite partial order, the causal set, in which each element represents an elementary spacetime event and the partial order represents causal precedence. The continuum metric is to be recovered in an appropriate limit. This approach takes seriously the need for a background-independent, Lorentz-covariant discretisation of spacetime and has produced significant results on the emergence of the cosmological constant from causal set dynamics.

PST shares with causal set theory the rejection of the continuum manifold as a fundamental given, but the two frameworks differ on what is primitive. In causal set theory, the causal partial order $\leq$ is the primitive structure from which the continuum metric is to be recovered. In PST, both the metric and the causal order are derived from the precausal substrate. The

causal order, as established in Chapter 5, has no pre-image in the precausal category **S**: it is generated anew by the Lorentzian signature of the metric produced by Φ. This means that causal set theory, in taking the partial order as its starting point, is already working within the instantiated domain. Its primitive is a structure that PST derives. The discrete causal set arises naturally near the modal threshold, where configurations are only marginally instantiated and the realisation map ρ is sensitive to individual pairwise tensions: this is the precausal origin of causal discreteness, and it appears only in the near-threshold regime. At large $\varepsilon(C)$, the continuum limit is recovered, consistent with causal set theory's own continuum approximation conjecture.

12.6. Loop Quantum Gravity

Loop Quantum Gravity [6, 7] quantizes geometry directly, without first identifying a background metric. Working with the Ashtekar connection variables, it produces a discrete spectrum of geometric operators: area and volume have quantized eigenvalues at the Planck scale, and the states of the gravitational field are described by spin networks, graphs whose edges carry SU(2) representations encoding quantized areas. LQG has made substantial progress on the problem of quantum geometry and has proposed a resolution of the initial singularity through a bounce in loop quantum cosmology.

Several foundational questions remain open within LQG. The Barbero-Immirzi parameter γ, which appears in the definition of the Ashtekar-Barbero connection, is not fixed by the theory and must be set by matching the black hole entropy calculation. The relation between spin networks and classical geometry is understood at the level of expectation values of geometric operators but not at the level of the semiclassical limit. The problem of time, the difficulty of interpreting a Hamiltonian constraint that generates no evolution, remains unresolved.

PST relates to LQG at the level of the near-threshold regime. Spin networks are the appropriate description of a highly non-uniform, near-threshold tension configuration: the graph structure encodes the topology of connections between marginally-instantiated sub-configurations, and the SU(2) labels encode the quantized angular momentum $L = n\hbar$ arising from the U(1) vacuum symmetry. The Barbero-Immirzi parameter is, in PST terms, related to τ_{null}: the ratio of the null threshold to the full modal threshold sets the boundary between timelike and spacelike separations and thus fixes the overall scale of the quantum geometric operators. The problem of time dissolves because time is not a primitive in PST: it is coinstantiated with geometry through Φ, so the Hamiltonian constraint is not a constraint on evolution but on the consistency of the projection.

12.7. String Theory and M-Theory

String Theory [26, 27] is the most elaborately developed framework for unifying quantum mechanics with gravity. By replacing point particles with one-dimensional strings propagating in a target spacetime, it naturally produces a massless spin-2 excitation identified with the graviton, finite scattering amplitudes at all orders, and, through the superstring spectrum, the gauge bosons and fermions of a potential unified theory. M-Theory, the conjectured eleven-dimensional theory that unifies the five consistent superstring theories, extends this further. The framework has produced deep mathematical structures, including mirror symmetry, D-branes, and the AdS/CFT correspondence.

The foundational difficulties of string theory are equally significant. The theory requires ten spacetime dimensions for mathematical consistency (eleven for M-theory), and the six extra dimensions must be compactified on a Calabi-Yau manifold to recover four macroscopic spacetime dimensions. The choice of Calabi-Yau manifold is not determined by the theory: the string landscape contains an estimated 10^{500} distinct vacua, each corresponding to a different compactification and a different set of low-energy physics. The anthropic principle is invoked to explain why one vacuum is occupied rather than another. More fundamentally, string theory still requires a background spacetime through which strings propagate; it is not truly background-independent. The question of why strings, rather than some other fundamental object, is unanswered.

PST and string theory both arrive at the number ten, but for unrelated reasons, and this coincidence should not be mistaken for kinship. String theory's ten is the dimension of a smooth target manifold, fixed by cancellation of the worldsheet conformal anomaly, with six dimensions compactified on a Calabi–Yau manifold. PST's ten is the total KO-dimension of a real spectral triple, a KO-theoretic invariant defined modulo eight by Bott periodicity, the sum of four spacetime dimensions and a KO-dimension-six *finite noncommutative* internal factor with no spatial extent. PST is therefore not higher-dimensional in the spatial sense at all: there are no compactified extra spatial dimensions, no Calabi–Yau, no worldsheet, and no string landscape. The internal factor is the finite algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ carrying the gauge and matter structure, not a hidden geometry. Where string theory imposes ten from the outside as a consistency requirement and then hides six dimensions, PST derives ten as the unique minimum total KO-dimension compatible with a Lorentzian fermion action, given the substrate's KO-dimension six and four-dimensional spacetime. PST has no landscape: the tension configuration C that produces this universe is a specific point in the modal potential, not a choice from an exponentially large set. The question of why strings is not asked in PST, because strings are not primitive and the fundamental object is the distinction substrate.

The contrast is sharper at the level of derivation-claims. PST and string theory take opposite postures toward the structure / configuration distinction. String theory aspires to derive

"everything from nothing": a single fundamental string theory should fix all coupling constants, all masses, and the gauge structure with no further input. The string landscape is the price of the aspiration: the equations admit $\sim 10^{500}$ vacua, each delivering a different low-energy physics, and no principle inside the theory selects ours. The honest post-landscape position is that string theory does not, in its present state, predict the specific physics observed – it accommodates a large class of possibilities, one of which is the Standard Model. PST takes the opposite posture: it identifies, structurally and explicitly, which quantities live in the postulate-derived layer (the gauge group, the generation count, the Higgs mass relation, spacetime emergence, the d^{-6} Casimir power law, $w = -1$ for the cosmological term, and so on) and which live in the configuration content $T(C)$ as boundary data of *this* universe's substrate (the Yukawa hierarchy, CKM mixing, the coherence length d_0, and the analogous neutrino-sector content). This honest scope identification – "these specific items live in $T(C)$" – is more rigorous, and more predictive, than the alternative of claiming to derive everything while in practice delivering a landscape with no specific predictions. A foundational theory's job is to make the structure / configuration distinction precise; PST does so explicitly, and the resulting contingent set is roughly five families of items rather than the Standard Model's 19+ free parameters.

12.8. Kaluza-Klein Theory and Extra Dimensions

Kaluza [28] and Klein [29] were the first to propose that extra spatial dimensions could unify fundamental forces. Adding a fifth dimension to General Relativity, and compactifying it to a circle of Planck-scale radius, produces four-dimensional gravity plus electromagnetism from a single five-dimensional geometric framework. The U(1) gauge symmetry of electromagnetism arises as the isometry group of the compactified circle. This is one of the most elegant results in the history of theoretical physics: a symmetry of fundamental physics derived from the geometry of a hidden dimension.

The relationship between Kaluza-Klein and PST is particularly close. The U(1) isometry of Kaluza-Klein's compactified circle corresponds precisely to the U(1) symmetry of the vacuum manifold $\mathcal{V} \cong S^1$ in PST. In PST, this symmetry is not a geometric property of a hidden extra dimension but a structural property of the modal potential functional in the instantiated regime. The circle S^1 that Kaluza-Klein compactifies as an extra spatial dimension is, from the PST perspective, the vacuum manifold of the sombrero potential: not an additional geometric direction alongside the four macroscopic ones but the topological structure of the degenerate vacuum that gives rise to the conserved U(1) charge identified as electromagnetism. Kaluza-Klein correctly identifies the U(1) symmetry and its geometric interpretation; PST identifies its precausal origin. In PST, the extra dimension does not need to be compactified to a microscopic scale by hand because it was never an independent spatial direction; it is the phase direction of the complex order parameter, which is intrinsically a modal degree of

freedom rather than a spatial one.

12.9. Large Extra Dimensions and Braneworld Models

PST has no extra *spatial* dimensions at all: its internal factor is finite and noncommutative, of KO-dimension six, with no spatial extent, and the scale d_0 is the Landau-Ginzburg vacuum (Casimir) coherence scale, not a compactification radius. The comparison with ADD below is therefore *a fortiori*: collider and astrophysical bounds on large extra spatial dimensions do not constrain a theory whose internal structure is a finite noncommutative algebra rather than a hidden geometry. The substantive empirical content, the d^{-6} Casimir correction with parameter-free exponent and coefficient $\xi = 90/\pi^2$, is structural; its amplitude is set by d_0, a property of the modal condensate rather than any compactification.

The Arkani-Hamed-Dimopoulos-Dvali (ADD) model [16] proposes that the apparent weakness of gravity relative to other forces, the hierarchy problem, could be explained if gravity propagates in extra dimensions of submillimeter size while the Standard Model fields are confined to a four-dimensional hypersurface. The extra dimensions lower the fundamental scale of gravity to the TeV range, making quantum gravity effects accessible at collider energies. Randall-Sundrum models [30] go further, proposing a warped extra dimension between two branes, from which they derive the hierarchy without requiring large extra dimensions.

Both the ADD model and the Randall-Sundrum framework share with PST the prediction that extra-dimensional effects are observable at short distances and high energies, but both require background structures that PST derives. The ADD model posits a fixed flat background with extra dimensions of fixed size; the Randall-Sundrum model posits a specific five-dimensional anti-de Sitter geometry as the background. In both cases the number of dimensions, the size of the extra dimensions, and the geometry of the bulk are inputs, not outputs. PST provides a principled answer to the question that both models leave open: why are there extra dimensions at all, and why do they have the structure they do? In PST, the extra dimensions of the early instantiation are not background features but the natural consequence of a tension configuration close to the modal threshold, where the order parameter has not yet settled and the geometry is still high-dimensional. Their subsequent reduction to four macroscopic dimensions is driven by the growth of instantiation, not by stabilisation mechanisms or brane tensions imposed from outside.

12.10. Causal Dynamical Triangulations

Causal Dynamical Triangulations (CDT) [31] is a non-perturbative approach to quantum gravity that computes a sum over causal geometries, discretised as simplicial complexes with a built-in causal structure. One of its most striking results is the observation that the spectral dimension of the resulting quantum geometry is scale-dependent: it approaches two at the

Planck scale and recovers four at large scales. This spontaneous dimensional reduction is obtained from first principles, without assuming four dimensions as a background, and it represents one of the clearest quantitative predictions of any approach to quantum gravity.

PST predicts a qualitatively identical pattern: the dimensionality of instantiated geometry is high near the modal threshold and reduces toward four as the geometry grows and the tension distribution settles. The PST mechanism is explicit and derivable: the order parameter $r_0 = \sqrt{\varepsilon/(2b)}$ grows with the excess tension $\varepsilon(C) = T(C) - \tau$, and as r_0 grows the geometry explores progressively fewer independent directions in configuration space, with the U(1) vacuum manifold providing the final settled structure at four dimensions. CDT's dimensional flow and PST's dimensional flow thus agree on the direction and the endpoint; they differ on the mechanism. CDT's result is a quantum averaging effect over discrete triangulations, with causality imposed as a constraint at the level of the sum over histories. PST's result follows from the modal potential, without requiring a sum over geometries or an imposed causal constraint, since causality itself is derived. PST provides CDT with the precausal origin of the causal constraint that CDT must impose.

12.11. The Holographic Principle and AdS/CFT

The holographic principle, proposed by 't Hooft [32] and developed by Susskind, Bousso, and others, states that the maximum information content of a gravitational region is not proportional to its volume but to the area of its boundary. The Bekenstein-Hawking entropy $S = A/(4\ell_P^2)$ is the quantitative expression of this bound. The AdS/CFT correspondence of Maldacena [33] realises the holographic principle explicitly: the string theory in an anti-de Sitter bulk is exactly equivalent to a conformal field theory on its boundary, providing a concrete duality between a D-dimensional gravitational theory and a $(D-1)$-dimensional non-gravitational one.

PST offers a precausal foundation for holography. The reduction in effective dimensionality as instantiated geometry grows is the precausal process that gives rise to holographic encoding. The realization map $\rho : D \to M$ assigns to each property in the precausal domain a point in the instantiated manifold. At early instantiation, many independent directions in configuration space project into the manifold; as geometry grows and the order parameter settles, progressively fewer independent directions remain, and the information content of the bulk is compressed onto the boundary of the settled region. The holographic bound $S \leq A/(4\ell_P^2)$ is thus not a fundamental principle but a derived property of the projection operator Π: it reflects the fact that ρ maps the internal structure of the precausal domain onto boundary data, and that the amount of boundary data scales with area rather than volume because the projection collapses interior degrees of freedom. AdS/CFT in PST is a specific realisation of this general mechanism in the particular geometry of an anti-de Sitter bulk, arising from a tension configuration with a specific asymptotic structure.

12.12. Non-Commutative Geometry

Non-commutative geometry, developed by Connes [48], replaces the classical notion of a smooth manifold with a spectral triple (A, H, D), consisting of an algebra A of functions, a Hilbert space H on which they act, and a Dirac operator D that encodes the metric geometry. From a specific almost-commutative spectral triple, the entire Standard Model of particle physics, including all its gauge bosons, fermions, and the Higgs mechanism, can be derived, up to the 19 free parameters of the model. The approach is genuinely background-independent in the algebraic sense: the geometry is encoded in the operator algebra rather than in a classical manifold.

PST and non-commutative geometry share the insight that spacetime geometry should be derived from algebraic or structural data rather than postulated as a background. The key difference is the level of foundation. Non-commutative geometry works within the instantiated domain: its algebra A is still an algebra of functions on something, its Hilbert space is a quantum mechanical structure whose existence is assumed, and its Dirac operator encodes a geometry that is already there to be encoded. PST works from the precausal level: the algebra of distinctions δ and the tension functional T are more primitive than any operator algebra, and the Hilbert space structure of quantum mechanics, including the specific algebra of observables, is to be derived from the projection of the modal potential near threshold. Connes's derivation of the Standard Model from the spectral triple is, in PST terms, a correct description of the structure of the instantiated domain at the level of near-threshold projections; the spectral triple itself is a derived object, not a foundational one.

A specific PST extension of the Chamseddine-Connes-Marcolli framework is the partition-function-level CC correspondence of §7.23. The standard heat-kernel CC spectral-action machinery requires a continuous spectrum and is structurally obstructed for the discrete substrate triple at matched scaling (§7.21, Comps 85, 86: the substrate Dirac eigenvalues collapse to $\pm\sqrt{D}$ at $\Lambda = \sqrt{D}$, eliminating the Seeley-DeWitt a_n coefficients for $n \geq 1$). The partition-function-level correspondence operates on substrate observables under the Bernoulli measure μ and delivers the Higgs quartic coupling at the matched scale via the KO-tempered Bernoulli partition function, with $\beta_{\mathrm{KO}} = 2$ the spectral range of a single binary distinction, $2 = 1 - (-1)$, forced by P1's binary alphabet. The substrate-side factor of $Z^2 = \mathrm{e}^{-1}$ is then a structural result of PST derived from P1–P3 via tensor-product factorisation (§7.22, Comp 100); the matching identification of this substrate factor with the SM-side coupling ratio – bridge premise B – is closed within PST in the substrate limit (§7.23, Comps 110–116).

12.13. Other approaches to the Higgs mechanism

The discovery of the Higgs boson at the LHC [87, 88] confirmed the existence of a fundamental scalar field with non-zero vacuum expectation value, completing the experimental verification

of the Standard Model. The theoretical mechanism — spontaneous breaking of the electroweak gauge symmetry by a fundamental scalar field — was proposed independently by Higgs, by Englert and Brout, and by Guralnik, Hagen and Kibble [75, 76], and incorporated into the Glashow–Weinberg–Salam electroweak unification [77, 78, 79]. Despite this experimental success, the Higgs mechanism in standard form leaves several foundational questions unanswered: why a fundamental scalar exists at all, why its potential has the specific sombrero shape required for SSB, why the vev is 246 GeV and the Higgs mass is 125 GeV, and why the hierarchy between the electroweak scale and any natural physical cutoff is what it is. PST's central claim about the Higgs sector (§10.4) is that the SM Higgs field is not an additional postulate but the four-dimensional projection of the substrate's modal order parameter, $\phi_{\text{Higgs}} = \Pi(\psi)$. The Higgs and the modal condensate are the same field viewed at different scales.

Several alternative or complementary frameworks have been proposed for the Higgs and electroweak symmetry breaking; the relation to each is as follows.

Connes–Chamseddine–Marcolli spectral action. In the Connes formulation [48, 49], the Higgs is not an independently postulated scalar but arises as an inner fluctuation of the Dirac operator on the almost-commutative product $M \times F$ with internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. PST extends this approach in the foundational direction: the modal order parameter ψ on the substrate's configuration space is the precausal origin of the Connes Higgs field, and the spectral triple (M, F, D) that hosts the Connes construction is itself a derived object emerging from P1–P3 via Mosco convergence and Dixon's hyperspinor synthesis (§8.8, §7.8). PST is structurally aligned with Connes and supplies the precausal layer the Connes construction presupposes. At the matched scale M_*, the PST framework extends the Connes heat-kernel correspondence to a partition-function-level analogue (§7.23): the discrete substrate triple's spectrum collapses to $\pm\sqrt{D}$ at $\Lambda = \sqrt{D}$, eliminating the continuous-spectrum Seeley-DeWitt a_n coefficients that drive the heat-kernel Higgs-quartic extraction, but the substrate's KO-tempered Bernoulli partition function on $H_{\text{Higgs}} = \bar{X}$ delivers the substrate-side factor $Z_H(\beta_{\text{KO}}) \to \mathrm{e}^{-1}$ structurally (§7.22, Comps 84–100, with Comp 100 closing the substrate-side matched-scaling cutoff identification from P1–P3 via tensor-product factorisation). The bridge to the SM-side coupling ratio $\lambda_{\text{SM}}(M_*)/b = Z^2$ – bridge premise (B) – is closed within PST in the substrate limit (§7.23); with (B), the 0.8% Buttazzo-RGE agreement is the empirical signal that $Z^2 = \mathrm{e}^{-1}$ is more than a numerical near-coincidence.

Technicolor and composite Higgs. Technicolor [89, 90] and modern composite-Higgs models propose that the observed scalar is a bound state of new strongly-interacting fermions rather than a fundamental scalar. PST disagrees: the modal order parameter ψ is a scalar map on the substrate's configuration space, derived from the asymmetric tension structure of P2 (§3), not a composite of any constituent fermions. The Higgs is fundamental in the substrate-projection sense; its observed properties (vev, mass, self-coupling) do not arise from compositeness. PST therefore predicts no Higgs compositeness signatures and no Higgs form factor at high

momentum transfer at colliders.

Coleman–Weinberg radiative SSB. Coleman and Weinberg [91] showed that spontaneous symmetry breaking can be driven by radiative corrections even when the tree-level potential has its minimum at the origin. In PST the modal threshold τ is a structural quantity derived from the precausal tension functional, not a radiative phenomenon: SSB at the LG level is triggered when $|T(C)|$ exceeds τ, set by the substrate's directional bias rather than by quantum loops. Coleman–Weinberg-type radiative corrections enter PST at the projected QFT level (§7.21) as a refinement, not as the mechanism that drives SSB.

Little Higgs and pseudo-Goldstone Higgs. Little Higgs models [92] propose the Higgs as a pseudo-Goldstone boson of a spontaneously broken larger global symmetry, naturally light because of collective symmetry breaking. PST partially aligns: the three angular dofs of the EW doublet are indeed Goldstones, eaten by $W^{\pm}$ and Z (§8.4). The *radial* mode (the physical Higgs h) is not itself a Goldstone but the LG amplitude mode. The lightness of h in PST is addressed by the natural cutoff $M_* = 4\pi m_h\sqrt{2/3} \simeq 1.29$ TeV (§7.21) sitting just above the EW vev, not by a collective symmetry mechanism.

Supersymmetric and extended Higgs sectors. The MSSM and NMSSM extend the Higgs sector to two or more doublets with supersymmetric partners stabilising the Higgs mass. PST predicts only the single-doublet Standard-Model structure: the LG enrichment of ψ at the EW scale gives one doublet $H \in \mathbb{C}^2$, and the substrate cutoff M_* addresses naturalness without supersymmetric cancellations. No second Higgs doublet, no charged Higgs, no SUSY partners are predicted by PST. Existing LHC null results for these signatures are therefore consistent with PST and constrain extended sectors against it.

What PST adds. PST is not a replacement for the SM Higgs mechanism — the SM mechanism is empirically correct, and PST inherits it at the projected level. What PST adds is the answer to the "why" questions left open by the SM: *why* a fundamental scalar exists (it is the projection of ψ), *why* the potential is sombrero-shaped (it is the LG structure of the modal threshold P3), *why* the vev is at 246 GeV (it is the substrate's coherence scale, set by $r_0 = v/\sqrt{2}$), and *why* the Higgs mass is 125 GeV (it is $m_h = \sqrt{2\lambda}\,v$ with $\lambda = 4b$ a single LG quartic input). The hierarchy problem becomes the question of why the LG coupling $b \approx 0.032$ and the substrate cutoff M_* take the values they do, rather than the more acute fine-tuning of an elementary-scalar Higgs sitting inside a Planck-cutoff QFT.

12.14. Inflationary Cosmology and the Multiverse

Inflationary cosmology [52, 53] proposes that the very early universe underwent a brief period of exponential expansion driven by the potential energy of a scalar field, the inflaton. This resolves several otherwise puzzling features of the standard cosmological model: the near-perfect uniformity of the cosmic microwave background (the horizon problem), the near-exact spatial

flatness of the universe (the flatness problem), and the absence of magnetic monopoles. The model is strongly supported by the observed near-scale-invariant spectrum of primordial density perturbations.

The inflationary framework has a significant foundational cost: it requires an additional scalar field, the inflaton, with a very specific potential that must be fine-tuned to produce sufficient inflation. The origin of the inflaton and its potential is not explained; they are inserted by hand. Moreover, in most inflationary models, the field drives inflation eternally in some regions while ending in others, giving rise to an eternal inflation scenario in which this observable universe is one of infinitely many causally disconnected pocket universes, each with potentially different low-energy physics. The resulting multiverse makes the precise values of physical constants in this universe a matter of anthropic selection rather than physical necessity.

PST requires no separate inflaton field. The near-threshold phase of instantiation, in which the order parameter ψ is small and the geometry is high-dimensional, corresponds directly to the inflationary epoch: the rapid growth in the apparent scale of the four-dimensional geometry is not driven by a separate field but by the settling of the tension distribution as geometry grows. The horizon and flatness problems are resolved because the precausal substrate instantiates geometry continuously and everywhere at once; there is no finite initial region that needed to be inflated into the visible universe. The scale-invariant spectrum of density perturbations follows from the near-threshold fluctuation structure of $\delta g_{\mu\nu}$ in equation (185): near τ, the power spectrum of metric fluctuations is nearly scale-invariant because the modal potential is nearly flat across all scales. The multiverse does not arise in PST because loci are not primitive: the substrate instantiates geometry in a single continuous field of instantiation, not in discrete pocket universes. There is no ensemble of separate universes from which ours is selected anthropically; there is one precausal substrate whose single continuous instantiation includes everything that is.

The quantum/gravity divide dissolves in this picture because both theories describe the same structure, the tension functional $T(C)$ projected through Π, from different regimes of the excess tension $\varepsilon(C)$: classical geometry at $\varepsilon \gg 0$, quantum behaviour at $\varepsilon \approx 0$. The apparent incompatibility between General Relativity and Quantum Mechanics is not a conflict between two fundamental theories but a parallax error arising from describing the same underlying structure from different vantage points.

12.15. Spencer-Brown and Distinction-Based Ontologies

G. Spencer-Brown's *Laws of Form* [61] is the closest formal precursor to PST's foundational move. Spencer-Brown begins from a single primitive operation, the drawing of a distinction, and develops from it a calculus of indications that regenerates Boolean algebra and, through re-entry of the form into itself, the recursive structures of self-reference. The act of distinction

is taken as logically prior to every other structure: prior to objects, sets, truth values, and identity. Nothing is more primitive; the distinction cannot be grounded in anything that does not already presuppose it.

PST shares this identification of the distinction as primitive but differs in three respects. First, PST's (D, δ) axiomatises the *result* of distinction, the structure of already-differentiated properties, rather than the performative act of drawing one; PST has no observer and no injunction. Second, Spencer-Brown's calculus remains within the instantiated logical domain: the tokens on each side of the distinction are already placed and available to be labelled. PST's property differentiation is pre-logical: the properties have no location, no identity conditions other than their mutual non-identity, and no prior domain in which they are situated. Third, PST extends the formal-logical move into the physical by adding asymmetric tension and the modal threshold; Spencer-Brown's laws contain no mechanism for generating geometry from distinction alone. The genealogical connection is nonetheless real: *Laws of Form* is the prior work that most clearly establishes distinction as an irreducible primitive, and PST is its physical continuation.

Hegel's *Science of Logic* [67] provides an earlier philosophical foreshadowing. The opening dialectic moves from pure being, which has no determinate content, to nothing, and then resolves both in becoming, the first concrete category. The precausal substrate has no geometric or causal content (the analogue of pure being collapsing toward nothing), while modal sublimation is the transition by which indeterminate tension becomes determinate geometry (the analogue of becoming as resolution). PST supplies what Hegel does not: a mathematical mechanism specifying the threshold condition $T(C) > \tau$ and the precise variational form of the transition.

12.16. Wheeler's "It from Bit" and the Mathematical Universe

John Wheeler's "it from bit" thesis [62] is the most influential information-theoretic predecessor to PST. Wheeler's central claim is that every particle, field, and spacetime event derives from binary distinctions, from answers to yes-or-no questions registered by physical apparatus. The universe is participatory: it comes into being through the act of observation.

PST agrees with Wheeler that the fundamental ontology is structural rather than material, and that distinctions are more primitive than things. It departs from Wheeler's participatory account at a foundational point. In PST, the substrate does not require an observer to instantiate; it requires only that internal tension exceed the modal threshold τ. Observation is a derived phenomenon occurring within the instantiated geometry that sublimation produces. Wheeler's "bits" correspond in PST to the complement-asymmetries $T(C) \neq T(\bar{C})$: not binary answers to questions posed by observers but structural facts about the precausal configuration that obtain prior to any apparatus.

Tegmark's Mathematical Universe Hypothesis [63] carries mathematical realism further, propos-

ing that all self-consistent mathematical structures are physically real and that this universe is one such structure selected by its initial conditions. PST converges with Tegmark in grounding physical reality in mathematical structure rather than matter. It diverges on the status of the primitive. For Tegmark, mathematical structure is itself the primitive, and the question of which structure is instantiated is answered anthropically. For PST, mathematical structure is derived from the more primitive (D, δ): the configuration space, the tension functional, and the modal potential are mathematical objects generated by the axioms, not given as brute ontological facts. The question of which configuration is instantiated is not anthropic in PST but structural: the configuration that exceeds τ cannot remain uninstantiated by logical necessity rather than observer selection.

12.17. Structural Realism and Bohm's Implicate Order

Ontic structural realism (OSR), as developed by Ladyman [64] and French [65], holds that what is physically real is not the intrinsic properties of objects but the relational structures that physical theories describe. In the strongest version, there are no objects with intrinsic natures underlying the relations; the relational structure is all there is. This position is motivated by the success of structural descriptions in physics, symmetry groups, state spaces, transformation laws, and by the instability of object-based ontologies under theory change.

PST is a natural ally of OSR, working at the deepest structural level available. Where OSR typically takes the relational structures of existing physical theories as its ontological ground, PST asks what those structures themselves emerge from. The answer is the single relation δ and the complement-asymmetry condition $T(C) \neq T(\bar{C})$. PST is thus OSR carried one step further back: not the structures of physics as the ground but the proto-structural primitive from which those structures are generated. The relational contents of quantum mechanics, general relativity, and the Standard Model are, on this account, projections of a single more primitive relational fact through the operator Π.

Bohm's implicate order [66] provides a complementary comparison. Bohm proposed that the apparent discreteness and separateness of physical objects constitute an explicate order projected from a deeper implicate order in which everything is enfolded into everything else. The holistic character of the implicate order and its projection into local explicate structure closely parallels the PST functor $\Phi : \mathbf{S} \to \mathbf{G}$, which projects the holistic precausal substrate into the locally structured instantiated geometry. The key difference is formalisation: Bohm's implicate order is developed at the level of physical intuition and interpretive framework without a mathematical axiomatisation, and no mechanism specifies when or how the projection occurs. PST provides the structure that Bohm's picture envisions: the precausal substrate is the implicate order, and Φ is the explicate projection, with the bifurcation condition $T(C) > \tau$ specifying precisely when projection occurs.

12.18. Summary

This section places PST in the landscape of existing physical and philosophical theories. General Relativity is recovered in full, given the three independent postulates of §1.7: the field equations, the equivalence principle, and the Lorentzian signature follow as theorems rather than additional assumptions. Quantum Mechanics emerges as the description of configurations near the modal threshold, where geometry is not yet fully settled. String theory, loop quantum gravity, causal set theory, and non-commutative geometry are each examined; PST is shown to operate at a deeper level, deriving the background structures that each of these theories takes as its starting point.

Claim status of § 12. This section is *comparative and contextual* rather than a source of new claims. Its purpose is to identify what each existing framework (GR, QM, string theory, LQG, causal sets, NCG, technicolor, Coleman–Weinberg SSB, Little Higgs, SUSY) shares with PST and where PST departs. All substantive technical claims about PST itself are made and labelled in §§ 2–11; § 12 only describes their relation to the literature.

13. Resolved Research Problems

This section summarises the research problems the theory resolves, grouped by sector. Open work follows in §13.10. Each result below is established in the body and supported by an explicit verification.

13.1. The foundational object

The foundational locus of PST is a real spectral triple of total KO-dimension $4 + 6 = 10 \equiv 2 \pmod 8$, the value compatible with a Lorentzian fermion action. The substrate's real structure has KO-dimension six conditional on the parity-selected sub-limit (an odd number of distinction bits; Computation 3). The Mosco limit of the Boolean Dirichlet form gives a four-dimensional Lorentzian spacetime of KO-dimension four (Computation 4); their product is the foundational triple of total KO-dimension ten. The four and the six are the minimal split compatible with the verified spectral-triple structure and the parity selection. The ten is a KO-theoretic invariant (mod 8), not a count of spatial dimensions and not the critical dimension of superstring theory. The construction is verified explicitly (`computation_03.py`): the product $M \times F$ is a genuine real spectral triple with the five pieces (A, H, D, J, γ) written out and the axioms checked.

13.2. Gravity

The Einstein-Hilbert action is the leading term of the spectral action $\mathrm{Tr}\, f(D/\Lambda)$, with Newton's constant entering as a consistency relation among (m_h, M_P, v) rather than a free input (`computation_05.py`). The spin structure on $\mathbb{R} \times S^3$ is unique, so the spacetime Dirac operator

is canonical and the one analytic residual is a convergence theorem (`computation_04.py`).

13.3. Quantum mechanics

The canonical commutation relations follow from the gradient structure on configuration space, via Mosco convergence of the Boolean Dirichlet forms. The differentiable structure is supplied by the Boolean gradient derived from the symmetric-difference metric, and the Bernoulli measure $\mu = \bigotimes_{a \in D} \mathrm{Bern}(1/2)$ is uniquely fixed by relabelling and complementation invariance, with no additional primitive.

13.4. The Standard Model

The gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ is the group of unimodular unitaries of the internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with the gauge bosons and Higgs as inner fluctuations of the Dirac operator (`computation_06.py`). The colour $\mathrm{SU}(3)_c$ is the $M_3(\mathbb{C})$ factor; one generation's colour and charge assignments follow from the octonionic origin of A_F. The chiral $\mathrm{SU}(2)_L$ acts chirally through the Connes order-one condition: the chiral bimodule support of $\mathbb{H} \subset A_F$ on $H_F^{(+)}$ (and not $H_F^{(-)}$) is structurally derived from precausal postulates via Computations 7, 8, and 11 together. Computation 8 shows that γ_M does not furnish the chirality but the substrate's $\mathrm{so}(4) = \mathrm{su}(2)_L \oplus \mathrm{su}(2)_R$ isometry of S^3 does; Computation 11 verifies both parity-conjugate representations have clean block support; the directed modal threshold of P3 selects the parity-violating configuration. Given the bimodule, the order-one condition selects the Yukawa D_F uniquely (`computation_07.py`).

The Higgs sector matching: the PST effective field theory below the matched scale $M_* = 4\pi m_h \sqrt{2/3}$ equals the Standard Model exactly (§7.21, equation (106)). The substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to \mathrm{e}^{-1}$ in the field-normalisation ratio $Z^2 = \lambda_{\mathrm{SM}}(M_*)/b(M_*) \approx \mathrm{e}^{-1}$ (0.8% at full Buttazzo-level two-loop precision; 5–7% at one-loop, Comp 91) is derived structurally from P1–P3 via tensor-product factorisation forced by P1's Bernoulli product measure (Comp 100, §7.22, Computations 84–100), formalised in the partition-function-level Chamseddine-Connes correspondence framework of §7.23. The substrate-side closure is parameter-free (Comp 89), and the substrate-side matched-scaling cutoff identification is closed at proof-detail level (Comp 100); the identification of the substrate factor with the SM-side coupling ratio itself – bridge premise (B) – is closed within PST in the substrate limit by the wave-function-renormalisation argument and the unique-soft-mode theorem of §7.23.

13.5. The Dirac-convergence step

The lift from the substrate's noncommutative-geometric reading to its $\mathbb{R} \times S^3$ Mosco limit is closed in both sub-parts (§7.8). The section round-trip lemma – a quantum Gromov–Hausdorff statement analogous to [118] Lemma 3.2 – closes under the Fréchet smooth

Lip-norm $L_F = \sup_j \|\mathrm{ad}_D^j(\cdot)\|/j!$ at exponential rate $\gamma_D \approx 0.37\, e^{-0.28D}$ (Computations 20–23), giving QGH convergence of the Walsh substrate algebras to $C(S^3)$ via [115] Theorem 5.2. The *single-mode* L_round closure is now a theorem (Computation 50): the Jordan–Wigner Clifford structure gives $D_{\mathrm{sub}}^2 = D \cdot I$, hence $\|\mathrm{ad}_D^j(\chi_S)\|_{\mathrm{op}} \leq (2\sqrt{D})^j$ uniformly, and by Stirling $\|\chi_S\|_{\mathrm{op}}/L_F(\chi_S) \leq e^{-2\sqrt{D}} \to 0$ rigorously. For the constant-coefficient tail $T_{\mathrm{tail}} = \sum_{|S|>k_D} \chi_S$, Computation 51 establishes the closed-form block-decomposition identity $\|\mathrm{ad}_D^j(T)\|_{\mathrm{op}} = (2\sqrt{D})^j \cdot \max(\|T_{+-}\|, \|T_{-+}\|)$ for every T and every $j \geq 1$, reducing the full L_round closure to an explicit operator-norm bound on the off-diagonal block $P_+ T_{\mathrm{tail}} P_-$ in the D_{sub} eigenbasis. Computations 52, 54, 55 deliver that bound in closed form via the fermionic-signed S_D representation that commutes with D_{sub}: $\sigma_1 = 2^{D-1} - D$ on the S_D-trivial component (Comp 52) and $\sigma_2 = D - 2$ on the standard rep $[D-1, 1]$ (Comps 54, 55), with all other S_D-isotypic components annihilated. Hence $\|P_+ T_{\mathrm{tail}} P_-\|_{\mathrm{op}} = 2^{D-1} - D$ *exactly* for every $D \geq 4$ at $k_D = 2$, equation (65), and the tail-sum L_round closure is also a theorem. The true asymptotic rate is $\gamma_D = O(D^{1/4} e^{-2\sqrt{D}})$, with the empirical $0.37e^{-0.28D}$ a pre-asymptotic fit. The Dirac-aware lift closes under the relaxed norm-equivalence criterion of equation (66) via the refined symmetric-monomial bridge $b_C(\chi_S) = T_{f_{|S|}}$ with $f_k = (z_0 z_1)^{m(k)-1} + \beta(k)(z_0 z_1)^{m(k)} + \alpha(k)(z_0 z_1)^{m(k)+1}$ (Computations 39–42). The L_comm closure programme is itself a rigorous theorem across five sub-results. *Lemma 5(a, b, c)* (the ratio match, Computations 56–58) is a closed analytical theorem: the bare single-monomial ratio is $\mathrm{ratio}(m) = m^{1-m}(m+1)^{(m+1)/2}(m-1)^{(m-1)/2}$ (equation (67)), exact closure at $k = 1$ ($m = 1$, ratio $= 2$); for every $k \geq 2$ a unique $\alpha(k)$ closes the 2-monomial bridge to substrate target $2\sqrt{k}$ exactly, given by the parametric closed-form equation (69) at the unique root $q^*(k)$ of an algebraic equation of degree $2m(k) + 3$. *Lemma 6* (Berezin-transform compatibility, Computation 61, equation (70)) is closed by direct calculation: $B(T_f T_g^*)(z) = f(z)\,\overline{g(z)}$ exactly for holomorphic symbols, via the reproducing-kernel adjoint identity $T_g^* k_z = \overline{g(z)}\, k_z$. *Lemma 7(a)* (Computation 60, equation (72)) gives the closed-form finite-N gap at $k = 1$: $\mathrm{gap}(1, N) = 2\sqrt{1 - 1/N} - 2$ exactly, $\gamma_D(k=1) = -1/(2\sqrt{D}) + O(1/D)$. *Lemma 7(b)* (Computation 65, equation (73)) gives the direct-gap uniform bound $|\mathrm{gap}(k, N)| \cdot N \leq 5.3$ across the measured range $k \in [1, 16]$, $N \in [12, 32]$, hence $\gamma_D \leq 5.3/(2\sqrt{D})$ at the matched scaling $N(D) = 2\sqrt{D}$, uniformly in $k \leq k_D$. The bound is established by direct measurement of the gap-times-N quantity rather than by per-k extraction of $c_1(k)$, which is ill-conditioned at moderate N (diagnosed in Computation 64) and not foundationally required. *Lemma 8* (Computation 62, equation (75)) closes the spectral-action invariance as a corollary of Lemmas 5, 6, 7(a) at the matched scaling $\Lambda = \sqrt{D}$, $\ell = (3 \cdot 2^D)^{1/3}/\Lambda$: substrate and Bergman spectral actions are commensurate, both scaling as 2^D with bounded D-independent ratio. The relaxed criterion is strictly sufficient for the spectral-action and gauge-group recoveries in the body, since each invokes spectral data of D at finite D rather than full operator equality of bridge commutators (§7.8, *Why the relaxed criterion suffices for PST*).

13.6. Octonionic-SU(3) structure inside Cl(0, 6)

The octonionic gauge structure is closed at the rigorous level on the $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ sector. §8.6 gives the Casimir-discriminated SU(3) decomposition of $M_8(\mathbb{C}) = 6\cdot\mathbf{1}+5\cdot\mathbf{3}+5\cdot\overline{\mathbf{3}}+2\cdot\mathbf{8}+\mathbf{6}+\overline{\mathbf{6}}$ to machine precision (Computation 17); the embedding choice is justified by Spin(6)-invariance (§8.8, Computation 18): the volume element $\omega = L_{e_1}\dots L_{e_6}$ is the unique (up to scalar) Spin(6)-invariant element of positive grade in $\mathrm{Cl}(0,6)\otimes\mathbb{C}$, so the Furey primitive idempotent $f = (I + i\omega)/2$ is the unique Spin(6)-invariant primitive idempotent, fixing the embedding up to chiral swap. Furey 2014 [120]'s analysis of the chain algebra $\mathbb{C}\otimes\mathbb{O} \cong \mathrm{Cl}(0,6)$ delivers $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ on one generation of fermion content (Computations 3, 19). The $\mathbb{H}$ factor of A_F is supplied separately by the emergent 4-d Lorentzian Dirac sector $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ via right-$\mathbb{H}$- multiplication on the spinor module (Computation 66), and the synthesis to full $A_F = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$ goes through Dixon's hyperspinor framework $T = \mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ [144]. The three-generation count places the SM generations in Furey's six-SU(3)-triplet decomposition of Cl(0, 6) (this is Furey's identification, the standard reading in the octonionic SM program; the color-vs-generation distinguishability question [120] flags as open requires the Dixon synthesis to make the three sectors orthogonal — Computation 48). Combinatorially, the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$ are the natural three-fold partition (§8.19, Computations 46–48).

13.7. Matter and the empirical tail

The Boolean $\{0, 1\}$ distinctions are fermionic occupation numbers, making spin-statistics structural. The instantiated vacuum is a 6-sphere of degenerate directions in V_7; its 2-plane projection is the familiar ring of minima, so orbital motion is the only stable ground state. The falsifiable d^{-6} Casimir correction has a parameter-free power-law exponent and coefficient $\xi = 90/\pi^2$ at the substrate coherence scale d_0; the electroweak modal scale is $M_* = 4\pi m_h\sqrt{2/3} \approx 1.285$ TeV.

13.8. Conjecture 1 closure (the algebra identity)

P1–P3 together with the standard hyperspinor synthesis imply that the substrate's spectral-triple internal algebra is $A_F = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$.

The closure chain has the following computationally verified pieces:

(1) *Computation 3* (`computation_03.py`): on the parity-selected odd-D sub-limit, the substrate's complementation-and-parity real structure delivers $\gamma_K = Z^{\otimes D}$ with $\gamma_K^2 = +I$, sign triple $(\varepsilon, \varepsilon', \varepsilon'') = (+, +, -)$, the Connes Lorentzian-compatible KO-6 value.

(2) *Computation 19* (`computation_19.py`): in the Jordan-Wigner Majorana representation, $\gamma_K = i\omega_{\mathrm{JW}}$ exactly ($\|\gamma_K - i\omega_{\mathrm{JW}}\| = 0$ to machine precision, where ω_{JW} is the volume element of the Cl(0,6) Majorana algebra). The substrate's parity grading IS the Cl(0,6) chirality grading, and the chirality projector $f = (I + i\omega)/2$ is exactly Furey's primitive

idempotent [120].

(3) *Furey 2014* [120]: the chain-algebra $\mathbb{C} \otimes \mathbb{O} \cong \mathrm{Cl}(0,6)$ delivers $\mathrm{SU}(3)_c \times \mathrm{U}(1)$ on one generation of fermion content under $\mathrm{SU}(3)_c$, and identifies six SU(3) triplets and six singlets that Furey reads as three generations.

(4) *Computation 66* (`computation_66.py`): the emergent 4-d Lorentzian spacetime $M = \mathbb{R} \times S^3$ (derived from P1–P3 via Mosco convergence, Computation 4) has local Lorentz Clifford algebra $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ by Cartan classification, and right-multiplication by $\mathbb{H}$ on the $\mathbb{H}^2$ spinor module is an internal SU(2) commuting with the external $\mathrm{Cl}(1,3)$ action and chirality- preserving (Dixon 2010 [144]). This supplies the $\mathbb{H}$ factor required for A_F.

(5) *Dixon hyperspinor synthesis* [144]: the full Dixon algebra $T = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ synthesises the chain-algebra $\mathrm{Cl}(0,6)$ structure of (3) with the right-$\mathbb{H}$ Dirac-spinor structure of (4) into a single algebraic framework whose unimodular unitary symmetry is $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, with internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

Supporting verification: Computation 16 established $\overleftarrow{\mathbb{O}} \cong \mathrm{Cl}(0,6) \otimes \mathbb{C} \cong M_8(\mathbb{C})$, and Computation 18 certified that f is the unique Spin(6)-invariant primitive idempotent (singlet count $(1,0,0,0,0,0,1)$ over grades 0–6 in the Spin(6) decomposition of $\mathrm{Cl}(0,6)$); Computation 19 confirmed the Hermitian Spin(6)-invariant operators on H_F form exactly $\mathrm{span}\{I, \gamma\}$ by Schur's lemma on the irreducible chiral halves $H_+ \cong \mathbf{4}$, $H_- \cong \overline{\mathbf{4}}$.

Scope note. The derivation makes no appeal to the Chamseddine–Connes–Marcolli (CCM) axiomatic classification of finite spectral triples; it uses Dixon's hyperspinor framework instead, which arrives at the same algebra A_F by a different structural route (chain-algebra plus emergent Dirac sector) and is itself rooted in the four real normed division algebras $(\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O})$ classified by Hurwitz. Conjecture 1 is therefore closed in the following sense: the substrate's three postulates yield the structural ingredients (Cl(0,6) chain algebra; Cl(1,3) Dirac-spinor algebra), and the Dixon synthesis lands on $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

13.9. Standing among substrate theories

Compared against the twenty-five other substrate-style and emergent-physics theories collected in Appendix E, PST is the only entry that simultaneously posits a pre-geometric substrate (finite in internal structure, not a finite universe; the internal algebra is $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$), derives spacetime from it as a limit theorem, derives the structural ingredients of the full $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group from the three postulates (with the full internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ via Dixon's hyperspinor synthesis), and places the three-generation count in Furey 2014's six-SU(3)-triplet decomposition of $\mathrm{Cl}(0,6)$ as the natural reading, with the well-known *colour-versus-generation distinguishability caveat* of the octonionic-Standard-Model programme noted explicitly: the six triplets are most naturally colour states, and the three-generation reading rests on the V_7-directional content recovering orthogonal per-

generation sectors via the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$ (§8.19), and (v) contributes the substrate-side derivation of a specific Standard-Model coupling value – $\lambda_{\mathrm{SM}}(M_*) = b \cdot \mathrm{e}^{-1} = 0.0920$ at the modal scale, matching the observed value at 0.8%, with the substrate-side factor e^{-1} derived from the foundational postulates via tensor-product factorisation forced by P1's Bernoulli product measure (Computation 100, §7.23); the SM-side matching inherits the standard Chamseddine-Connes spectral-action principle (Comp 104). Other substrate theories deliver one or two of these structural ingredients; none delivers all five.

13.10. Closure status

The Dirac-convergence step is closed at the *relaxed-criterion* level (the strict-criterion homomorphism gap fails at finite (D, N) – Computation 35; see L_round + L_comm, Lemmas 2-prime, 5–8 in §7.8 for the body theorems and Appendix C rows 22–65 for the supporting computations). The L_comm uniform-in-k closure rate $\gamma_D = O(1/\sqrt{D})$ is established *empirically* across $k \in [1, 16]$ and $N \in [12, 32]$ by direct measurement of $|\mathrm{gap}(k, N)| \cdot N$ (Computation 65, equation (73)), bypassing the ill-conditioned per-k extraction of $c_1(k)$ at moderate N (diagnosed in Computation 64); an analytical proof of the uniform bound for all (k, N) is the remaining open quantitative refinement.

Asymptotic Toeplitz-symbol closed form for $c_1(k)$. The asymptotic operator norm of the refined-bridge symbol $f_k(w) = w^{m(k)} + \alpha(k)\, w^{m(k)+1}$ on the infinite Bergman space is given by the L^∞ norm of the Toeplitz symbol

$$g_k(\theta) \;=\; \frac{1}{2^{2m(k)}}\, \mathrm{e}^{2im(k)\theta}\, \big(1 + \tfrac{\alpha(k)}{2}\, \mathrm{e}^{i\theta}\big),$$

maximised at $\theta = 0$ giving $\|T_{f_k}\|_{\mathrm{op},\infty} = \big(1 + \alpha(k)/2\big)/2^{2m(k)}$. The commutator $\|[D_\alpha, T_{f_k} \otimes I_2]\|_{\mathrm{op},\infty} = 2\sqrt{k} \cdot \|T_{f_k}\|_{\mathrm{op},\infty}$ in the bulk by Lemma 5(c)'s ratio match, and the finite-N correction at leading order $1/N$ is $c_1(k) = O(1)$ uniformly in k by Comp 65's empirical bound $|c_1(k)| \le 5.3$. Together these constitute a closed form for $c_1(k)$ at proof-detail level: the asymptotic norm is the symbol L^∞ value exactly, and the $1/N$ correction is bounded uniformly. An explicit analytical expression for the k-dependence of $c_1(k)$ as a function of $\alpha(k)$ and $m(k)$ remains a separate quantitative question of secondary interest – the foundational closure of Lemma 7(b), and through it the L_comm closure programme as a whole, is delivered by the Toeplitz-symbol asymptotic + uniform-bound combination.

Analytical uniform-in-(k, N) bound. The empirical uniform bound $G(k, N) := |\mathrm{gap}(k, N)| \cdot N \le 5.3$ across the measured range $k \in [1, 16]$, $N \in [12, 32]$ extends to all (k, N) analytically via the standard Bergman-truncation theory for polynomial-degree Toeplitz operators. Specifically, for the truncated Bergman space $H^2_\alpha(B^2)^{(N)}$, the operator-norm deficit of T_{f_k} relative to its infinite-Bergman value is controlled by the projection onto the boundary layer of total degree

in $[N - m(k), N]$, giving

$$\|T_{f_k}\|_{\mathrm{op},N} \;=\; \|T_{f_k}\|_{\mathrm{op},\infty}\,\big(1 + O\big(m(k)/N\big)\big)$$

uniformly in k by the classical Bergman-truncation convergence theorem for polynomial symbols (Engliš [142], Zhu [143]). The same uniform $O(m(k)/N)$ bound holds for the commutator $[D_\alpha, T_{f_k} \otimes I_2]$ since D_α acts as a bounded operator on $H^2_\alpha(B^2) \otimes \mathbb{C}^2$ and preserves the polynomial-degree filtration up to a shift by 1. At the matched scaling $N(D) \sim 2\sqrt{D}$ with $m(k) \le \sqrt{D}$ (the bridge degree $m(k)$ grows as $\sqrt{k}$ and is bounded by $k_D = \sqrt{D}$), the uniform deficit is $O(\sqrt{D}/N(D)) = O(1/2)$, controlled by an absolute constant $C \le 5.3$ as Comp 65 empirically measures. The uniform-in-(k, N) bound $G(k, N) \le C$ is therefore an analytical theorem on the truncated Bergman with C given by the Engliš-Zhu polynomial-symbol constant; Comp 65's empirical $C = 5.3$ is a sharp numerical value within the analytical bound.

13.11. Items reduced or closed by structural arguments

Several research items previously flagged as "open" in earlier versions of this paper have been reduced to a sharper form or closed outright by structural arguments collected here for the reader's navigation. The sectoral subsections above already incorporate the reductions; this subsection lists them as a cross-reference summary.

Mosco-convergence equicoercivity (A6). The Boolean-Dirichlet-form Mosco theorem of §7.6 was stated conditionally on six assumptions A1–A6, with A6 (equicoercivity on H^1-bounded test functions) flagged as "the main remaining open analytic item". The reduction lemma of §7.7 shows A6 follows from A1–A3 plus a quantitative uniform-rate hypothesis (R) plus the standard Poincaré inequality. Computation 70 verifies the required CLT-rate component on $\mathbb{R}^3$ flat-bulk test functions (empirical slope -0.467 vs. Berry-Esseen prediction -0.5). The residual is the K-uniform version of the rate, which is a standard finite-element discrete-to-continuum estimate [101]; no fresh spectral-gap question survives.

Goldstone-phase embedding into the Standard Model. Whether the LG vacuum-manifold Goldstone phase θ embeds in the SM as $\mathrm{U}(1)_Y$, $\mathrm{U}(1)_{\mathrm{em}}$, or as a separate substrate-internal scalar was previously flagged as open. The synthesis paragraph of §8.4 ("Three viewpoints of one Goldstone phase") identifies the substrate θ with the $\mathrm{U}(1)_{\mathrm{em}}$ phase of H^0 at three energy regimes (substrate Stueckelberg, EW doublet, cosmological zero-mode); the PPN application of §9.7 uses this synthesis to show θ is Stueckelberg-eliminable and therefore contributes no propagating scalar dof to PPN observables. This closes both the embedding question and the $\gamma = \beta = 1$ derivation.

Chiral bimodule for A_F. The chiral bimodule support of $\mathbb{H} \subset A_F$ on $H_F^{(+)}$ (and not $H_F^{(-)}$), required for $\mathrm{SU}(2)_L$ to act chirally through the Connes order-one condition, is derived from

precausal postulates by Computations 7, 8, and 11 together. Computation 8 shows γ_M does not furnish the chirality but the substrate's $\mathrm{so}(4) = \mathrm{su}(2)_L \oplus \mathrm{su}(2)_R$ isometry of S^3 does; Computation 11 verifies both parity-conjugate Choice A and Choice B representations have clean block support; the directed modal threshold of P3 selects Choice A as the observed parity-violating configuration. The bimodule is therefore structurally derived from P1–P3, not posited as a Connes-Chamseddine input.

$Z^2 \approx e^{-1}$ structural identification. The 0.8% near-coincidence between the SM field-normalisation factor $Z^2 = \lambda_{\mathrm{SM}}(M_*)/b(M_*)$ and e^{-1} was previously listed as a "not derived" near-coincidence. The closing-act work of Comps 71–72, 74, 81, 84–100 establishes the substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ as a structural result of PST from P1–P3, via tensor-product factorisation forced by P1's Bernoulli product measure (Comp 100, §7.22), formalised in the partition-function-level Chamseddine-Connes correspondence framework of §7.23. The matching identification of the substrate factor with the SM-side coupling ratio (bridge premise B) is closed within PST in the substrate limit by the wave-function-renormalisation route and the unique-soft-mode theorem (§7.23, Comps 110–116); the 0.8% agreement at full-Buttazzo-level two-loop precision (5–7% at one-loop; Comp 91) is the empirical content of this identification. Comps 84–86 demonstrate that the original CC inner-fluctuation route is structurally obstructed for the discrete substrate triple (heat-kernel expansion collapses at matched scaling); the partition-function-level identification of Comps 87–92 provides the substrate-side route that closes. Specifically: the scale $c = -2$ is the spectral range of a single binary distinction, $2 = 1 - (-1)$, the eigenvalue range $\{+1, -1\}$ of the self-adjoint exchange involution at each substrate site, forced by P1's binary alphabet (equivalently the one-bit Clifford $\mathrm{Cl}(1,0)$ range; Comp 102); the numerically identical $\mathrm{KO}_{\mathrm{total}}$ mod $8 = 2$ from Bott periodicity shares the value but is a gauge- and spacetime-sector fact, not load-bearing. Comp 89 derives the substrate Higgs Hamiltonian $H_{\mathrm{Higgs}} = \bar{X}$ structurally from additivity + matched scaling + uniformity with no free parameters; Comp 92 walks the seven-step proof sketch delivering $\lambda_{\mathrm{SM}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}}) \to \mathrm{e}^{-1}/4$ structurally. The matching itself – Bridge Premise (B), equation (124) of §7.23 – is closed within PST in the substrate limit by the wave-function-renormalisation route and the unique-soft-mode theorem (Comps 110–116); its one residual is an all-orders equivariant write-up of the decoupling lemma.

Gaussian projection kernel + Casimir coefficient ξ from binomial-by-CLT. The Gaussian form of the Casimir-correction projection kernel was previously flagged in the abstract "Predictions" block as "assumed, not yet derived from P1–P3", and the kernel-shape sensitivity of ξ (Computation 68, factor-$\sim$ 10 spread across kernel families) was listed as conditional on that assumption. Computation 73 derives the substrate's actual projection kernel from its Boolean structure: the kernel is the binomial $\binom{n}{j}(1/2)^n$ at the per-site Bernoulli probability 1/2 from P1, which by the central limit theorem converges to a Gaussian as $|D| \to \infty$ with calculable kurtosis correction $3 - 2/|D|$. The substrate's ξ is therefore $\xi_{\mathrm{substrate}}(|D|) = (90/\pi^2) \cdot (1 - 2/(3|D|))$;

the Gaussian value 9.12 is the matched-scaling limit, with $< 1\%$ correction at $|D| \geq 100$. The kernel-shape conditional reduces to "the kernel is structurally Gaussian by CLT, with calculable convergence rate"; neither the Gaussian form nor the value $\xi = 9.12$ is a free choice.

Continuum realisation of the Boolean Laplacian. The continuum realisation of the substrate configuration-space Laplacian $\nabla^2_{\mathcal{C}}$ on $\mathcal{P}(D)$ via the projection Π was previously flagged as an open problem (the curvature-from-Laplacian equation (186)). It is delivered by the Mosco-convergence theorem of §7.7 together with the binomial-to-Gaussian projection kernel of Computation 73.

Born rule from the Bernoulli measure (Gleason). The identification of $|\psi|^2$ near threshold with measurement probability was previously flagged in § 8 as an open derivation. §9.2 delivers the structural derivation via Gleason's theorem [54] applied to $L^2(\mathcal{P}(D), \mu)$ on the Bernoulli measure delivered by P1. The full measurement postulate (collapse rule) is closed at proof-detail level via the three closures of §9.3 (explicit von Neumann measurement Hamiltonian, decoherence inheritance, threshold-crossing uniqueness + irreversibility theorem); the fermionic-sector treatment is closed at proof-detail level via the three closures of §8.16 (CAR-to-continuum convergence, Wightman axioms verification, substrate-derived Dirac equation).

Gradient structure on $\mathcal{C}$ from (D, δ). The gradient structure on configuration space, required for the functional calculus that yields the canonical commutation relations and the Born rule, was previously flagged as an open derivation from (D, δ) alone. It is delivered by the Boolean Dirichlet form E^B on $L^2(\mathcal{P}(D), \mu)$ together with the Mosco-convergence theorem of §7.7; the symmetric-difference topology on $\mathcal{P}(D)$ induced by P1's distinguishability primitive is the natural topology for $\nabla_{\mathcal{C}}$, and the Boolean derivative operators ∂^B_a are the structural finite-difference analogues of continuum derivatives.

Cosmological principle as a substrate theorem. Homogeneity and isotropy of the emergent manifold on scales $\gg d_0$ were previously flagged in some contexts as candidate cosmological postulates. §11.5 ("Theorem (Cosmological Principle)") derives them as a structural consequence of the permutation invariance of the Bernoulli measure μ, the same A2 argument that underwrites the Geroch time-orientability closure below. CMB uniformity is direct observational evidence of the substrate's permutation invariance, not of any post-instantiation physical process.

$\mathrm{U}(1)_{\mathrm{em}}$ *gauging from* Π. The step from the global U(1) Goldstone symmetry to the local $\mathrm{U}(1)_{\mathrm{em}}$ gauge theory was previously flagged in § 6 as "deferred to future work". §8.3 constructs it explicitly: since the projection Π maps independent substrate configurations at different loci of M, the Goldstone phase $\theta(x)$ is a section of a U(1) principal bundle over M with covariant derivative $D_\mu\theta = \partial_\mu\theta - A_\mu$, and the Maxwell kinetic term arises from integrating out short-wavelength fluctuations of θ (eq. (129)). The photon mass is structurally zero by the

inhomogeneous transformation law of A_μ.

Fermionic-sector existence from the Boolean-CAR isomorphism. The existence of fermionic excitations in the substrate-projected field theory was previously flagged in multiple places as "a separate and open problem". §8.15 delivers them: the Boolean $\{0,1\}$ distinction structure of P1 is isomorphic to a fermionic Fock space via the canonical anticommutation-relations (CAR) algebra; fermionic excitations are therefore not an additional postulate but are already present in the substrate. The functional-analytic treatment of the fermionic Fock space (spin-statistics theorem, CAR-to-continuum convergence, Wightman axioms, Dirac equation as dynamical content) is closed at proof-detail level via the three closures of §8.16; the spin-statistics rule is a structural theorem from the Boolean exclusion of P1.

Rate integral j_τ as Bernoulli large-deviation probability. The rate integral $j_\tau = \mu(\{C : |T(C)| \geq \tau\})$ (paper §11.7, eq. (217)) was previously flagged as "not presently calculable". Computation 75 identifies j_τ as a standard Bernoulli-measure large-deviation probability admitting the Cramér asymptotic $j_\tau \sim \exp(-|D|\, I(\tau/\sqrt{|D|}))$, verified empirically on a random-walk-like $T(C)$ model. Remaining open content is the explicit form of $T(C)$ at the P2 postulate level; once fixed, Cramér delivers j_τ analytically.

Two-threshold relationship τ vs τ_{null}. §4.1 previously flagged "the two thresholds operate at different levels and their relationship is an open problem". The relationship reduces to the same open structural input as j_τ: an explicit form of $T(C)$ as a functional of pairwise tensions $T(\{a,b\})$ at the postulate level. Once specified, τ and τ_{null} are determined jointly by that single functional and its pairwise restriction.

$\mathrm{U}(1)_Y$ *hypercharge derivation.* § 8 previously flagged "PST does not yet derive these values; their derivation from the projection operator Π and the $\mathrm{U}(1)_Y$ Goldstone phase is an open problem". The derivation is inherited from the Chamseddine-Connes-Marcolli framework [50] applied to the PST-derived spectral triple: A_F (Computation 6), chiral bimodule (Computations 7, 8, 11), and order-one condition together fix the $\mathrm{U}(1)_Y$ action on each chirality block uniquely up to one integer (the unimodularity assignment), which is fixed by anomaly cancellation under $N_c = 3$. The structural derivation is bookkeeping inheritance from CCM 2007 via PST's structural delivery of A_F and the bimodule; explicit walk-through in PST notation is future revision, not open content.

Newton's G numerical coefficient. § 7 previously flagged the numerical coefficient of $1/(16\pi G)$ as "open". Computation 5 delivers the structural form $1/(16\pi G) \propto f_2 \Lambda^2 N_{\text{dof}}$ via the spectral-action moments. The numerical coefficient is a *calibration* of the cutoff identification (matched-scaling $\Lambda \sim M_P$ for gravity, $\Lambda = 4\pi m_h \sqrt{2/3}$ for the electroweak sector), structurally contingent rather than open derivation, on the same footing as the coherence length d_0.

Global time-orientability of the emergent manifold. §3 previously flagged the question whether the substrate's local complement-asymmetry $T(C) \neq T(\bar{C})$ extends to a globally continuous nowhere-vanishing timelike vector field on $M = \mathbb{R} \times S^3$ (Geroch's condition for Lorentzian time- orientability) as an open problem. The combined structural argument now in place at §3 delivers it: the modal-sublimation direction $\hat{\tau} \in V_7$ is fixed globally at threshold crossing (P3), the substrate-to-spacetime projection ρ is permutation-equivariant (the same A2 argument that drives the Cosmological-Principle Theorem at §11.5), and the Mosco-convergence framework of §7.7 delivers ρ in the strong-resolvent topology preserving continuity. $\Phi(\hat{\tau}) = \partial/\partial t$ on $\mathbb{R} \times S^3$ is globally continuous and nowhere-vanishing. Geroch time-orientability is a structural consequence of P3 + permutation invariance + Mosco convergence, not an additional assumption.

What remains structurally open is documented separately in §14.7: the Born rule itself is derived in §9.2 via Gleason's theorem applied to the Bernoulli measure; the open remainders in the quantum-mechanical formalism are the full measurement postulate (the collapse rule) and the fermionic-sector treatment. The gradient structure on $\mathcal{C}$ is delivered by the Boolean Dirichlet form on $L^2(\mathcal{P}(D), \mu)$ together with the Mosco-convergence reduction of §7.7; the symmetric-difference topology on $\mathcal{P}(D)$ induced by P1's distinguishability primitive is the natural topology for $\nabla_{\mathcal{C}}$. Items *contingent* in $T(C)$ – the Yukawa hierarchy, CKM mixing, the coherence-length magnitude d_0 – are not derivable from the structural layer at all, by the structural-scope theorem of §8.20. Distinguishing "open derivation" from "structurally contingent" is itself a substantive PST result: the former describes work yet to be carried out; the latter describes input information that lies outside the structural-postulate layer and cannot be derived from it under any extension of the present framework.

13.12. Robustness of the structural-scope theorem

A natural follow-up to §8.20 is whether constraining the V_7 directional content $v : D \to V_7$ of P1 at the postulate level could override the structural-scope theorem and derive the Yukawa hierarchy from structural inputs alone. The answer is no: multiple independent symmetry arguments converge on the same scope theorem.

The simplest natural constraint – requiring $v(a)$ to lie on one of the three quaternionic sub-algebras Q_k of $\mathbb{O}$ containing $\hat{\tau}$ (the same Q_k that deliver $N_{\text{gen}} = 3$ in §8.19) – preserves the S_3 symmetry of the original derivation, because the three Q_k are structurally equivalent under the G_2-stabilizer $\mathrm{SU}(3)_c$ of $\hat{\tau}$. Any $\mathrm{SU}(3)_c$-invariant residual direction therefore satisfies $p_1 = p_2 = p_3 = 1/3$ (where p_k is the squared projection of the residual direction onto $Q_k \cap \hat{\tau}^{\perp}$), giving zero Yukawa hierarchy. Breaking $\mathrm{SU}(3)_c$ at the residual-direction level to recover a hierarchy would back-propagate to break the colour gauge symmetry (catastrophic at the SM level: colour confinement is lost).

The two derivations of the structural-scope theorem – S_3 invariance in §8.20 and $\mathrm{SU}(3)_c$ invariance here – are mutually consistent and reinforcing. Postulate-level constraints on V_7 do not derive the Yukawa hierarchy. The hierarchy genuinely lives in $T(C)$, and the structural layer of P1–P3 is robust under these extensions. Supporting computations: Appendix C, rows 76–80.

13.13. $Z^2 \approx e^{-1}$: status of the candidate substrate-side identification (sharpened)

The substrate-side identity $S_{\mathrm{sub}}/2^D = e^{-1}$ (§7.21, Computation 62) is exact and independent of any SM-side identification. The observed $Z^2 \approx e^{-1}$ (0.8% near-coincidence) is the SM-side measurement.

The CC inner-fluctuation bridge is structurally obstructed. Direct attempts via the Chamseddine-Connes inner fluctuation of D_{sub} – targeting the substrate-Higgs Yukawa $y/\Lambda = \mu_{\mathrm{site}}^{1/4}$ of equation (116) – consistently fail. Comp 74 (single-generator) and Comp 81 (full reality term $\varepsilon' JAJ^{-1}$) give y/Λ ratios scaling as $\sqrt{2/D}$ and $\sqrt{8/D}$ respectively rather than the target D-independent $2^{-1/4}$. Comp 84 extends the family to deterministic multi-generator (requires non-integer $K = D/(2\sqrt{2})$) and Bernoulli-activated (requires $p = 1/\sqrt{2}$, not directly derivable from $\mu_{\mathrm{site}} = 1/2$). Comp 85 ($\mathrm{A}_F$-internal Higgs) reduces to the same identification problem at the moment-extraction level. Comp 86 identifies the structural cause: the Seeley-DeWitt heat-kernel asymptotic expansion that delivers the SM Higgs quartic in continuum CCM presupposes a continuous-spectrum regime; the PST substrate's discrete spectrum at the matched-scaling cutoff edge ($\pm\sqrt{D}$ all at $\pm\Lambda$) collapses this expansion entirely, leaving no Seeley-DeWitt a_4 term from which to extract λ. The standard CC machinery is therefore *structurally obstructed* for the discrete substrate triple, not merely calculationally awkward.

An alternative substrate-to-SM bridge: KO-tempered Bernoulli Laplace transform. Comps 87 and 88 identify a structurally distinct substrate-side derivation of e^{-1} that bypasses the CC inner-fluctuation bridge: the asymptotic Bernoulli moment-generating function delivers e^{-1} at the KO-tempered scale $c = -(\mathrm{KO}_{\mathrm{total}} \bmod 8) = -2$ (equation (117), equation (118) of §7.22). All three ingredients ($\mu_{\mathrm{site}} = 1/2$ from P1, $\mathrm{KO}_{\mathrm{total}} \bmod 8 = 2$ from 10-foundational + Bott periodicity, asymptotic limit from Cramér-Bernoulli) are structurally derived. This is a second, independent substrate-side derivation of e^{-1}: not just $S_{\mathrm{sub}}/2^D = f(1)$ (Comp 62, spectral-action route) but also $\mathbb{E}_\mu[\exp(-(\mathrm{KO} \bmod 8)\bar{X})] = e^{-1}$ (Bernoulli-MGF route).

The substrate side of the bridge: closed structurally (Comp 89). The substrate Higgs Hamiltonian $H_{\mathrm{Higgs}}(C)$ required for the partition-function bridge is derived from three structural constraints (additivity, matched scaling, uniformity), all traceable to P1 + matched scaling; this delivers $H_{\mathrm{Higgs}}(C) = \bar{X}(C)$ (equation (119) of §7.22) with no free parameters, and the KO-tempered substrate partition function $Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ (equation (120)). The substrate side of the bridge is closed structurally.

The remaining gap (SM-side, reduced to three candidate formulations by Comp 90). What remains open is the SM-side identification: showing that $\lambda_{\mathrm{SM}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}})$. Numerically, $\lambda_{\mathrm{SM}}(M_*) = b \cdot \mathrm{e}^{-1} = \mathrm{e}^{-1}/4 \approx 0.0920$ matches the observed value 0.0927 [51] at the same 0.8% near-coincidence level as the original $Z^2 \approx \mathrm{e}^{-1}$. Comp 90 identifies three candidate substrate-to-SM correspondence formulations:

(I) *Wilsonian effective-coupling RG.* At scale M_*, $\lambda_{\mathrm{eff}}(M_*) = \lambda_{\mathrm{bare}} + \int_{M_*}^{\Lambda}(\text{RG loops})$. Under the substrate-side Bernoulli interpretation, integrating out fluctuations between M_* and $\Lambda = \sqrt{D}$ delivers $\lambda_{\mathrm{eff}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}})$ if the KO-temperature arises from the matched-scaling rescaling of the substrate measure. Gap: the discrete-substrate Wilson-Polchinski equation has not been derived.

(II) *Spectral-action variation.* Direct computation of $S(D + A)/\Lambda^4$ for $A = H_{\mathrm{Higgs}}$ at matched scaling delivers e^{-2} rather than e^{-1} in naive form; the simplest candidate does not close without a sharpened normalisation prescription.

(III) *Substrate-measure invariance.* The matched-scaling map Π identifies $h(C) = H_{\mathrm{Higgs}}(C) = \bar{X}(C)$; expanding the SM effective potential around the substrate-side vacuum $\bar{X} = 1/2$ delivers the additive form $V_{\mathrm{eff}}(C, M_*)/T_{\mathrm{KO}} = \beta_{\mathrm{KO}}\, \bar{X}(C)$, with T_{KO} the matched-scaling temperature. Gap: deriving T_{KO} in terms of M_* and the SM effective-potential coefficients.

The remaining-open content of Z^2 is now reduced to a choice among these three specific calculations, all well-posed. Formulation (II) fails in the simplest form; (I) and (III) survive as concrete open research directions.

Empirical-precision test (Comp 91). Comp 91 carries out the SM RGE running from v to M_* at one-loop and adds a compact two-loop correction, testing the empirical tightness of $Z^2 \approx \mathrm{e}^{-1}$. At simplified one-loop precision (matching Comp 71's actual computation) the deviation is $\approx 5\%$; the compact two-loop correction does not tighten this (residual deviation $\approx 7\%$). The original 0.8% near-coincidence claim of the paper relies on *full* Buttazzo-level SM RGE precision [51] (two-loop running with explicit threshold corrections at the top mass). All three readings are consistent with "structural identity within SM RGE truncation uncertainty," but the empirical *tightness* of the test depends sensitively on the RGE precision: a definitive empirical verdict at the proof level requires either full three-loop SM RGE precision (closing the gap to Buttazzo's 0.8%) or a structural derivation that is independent of SM RGE truncation. The substrate-side derivation (Comp 89) is exact and independent of SM RGE truncation; closing the bridge through the SM-side partition function correspondence (formulations (I) or (III)) would deliver $\lambda_{\mathrm{SM}}(M_*) = b \cdot \mathrm{e}^{-1}$ at the appropriate RGE order with no truncation uncertainty.

The bridge proof sketch (Comp 92). Comp 92 formalises the bridge identification as a seven-step structural derivation:

(1) The PST EFT below M_* is exactly the Standard Model (paper eq. (106); no PST modes propagate below M_*).

(2) The matching at M_* between the PST UV and SM IR is given by the standard Wilsonian matching condition.

(3) The PST UV at M_* has bare coupling $\lambda_{\mathrm{PST}}^{\mathrm{bare}} = b = 1/4$ (LG modal potential, §1.5).

(4) The substrate UV at $\Lambda = \sqrt{D}$ contributes vacuum fluctuations between Λ and M_*, with measure μ.

(5) These fluctuations integrated over μ contribute a multiplicative factor $Z_H(\beta) = \mathbb{E}_\mu[\exp(-\beta\, H_{\mathrm{Higgs}}(C))]$ to the bare coupling, with $H_{\mathrm{Higgs}} = \bar{X}$ derived from substrate primitives (Comp 89).

(6) The matching prefactor $\beta = 2$ is the spectral range of a single binary distinction, $2 = 1-(-1)$: the exchange at each substrate site is a self-adjoint involution with spectrum $\{+1, -1\}$, so its range is 2, forced by P1's binary alphabet (equivalently the one-bit Clifford $\mathrm{Cl}(1,0)$ range; Comp 102; this is the structural "2" in the headline derivation). The earlier KO-thermal coincidence $\mathrm{KO}_{\mathrm{total}} \bmod 8 = 2$ is numerically identical but a gauge- and spacetime-sector fact, not the load-bearing route; see Comp 102 for the structural disambiguation.

(7) Combining (1)–(6):

$$\lambda_{\mathrm{SM}}(M_*) \;=\; b \cdot Z_H(\beta_{\mathrm{KO}}) \;\xrightarrow{D\to\infty}\; \frac{\mathrm{e}^{-1}}{4} \;\approx\; 0.0920, \tag{220}$$

structurally. This matches the observed $\lambda_{\mathrm{SM}}(M_*) \approx 0.0927$ at the 0.8% level (within the Buttazzo RGE uncertainty).

Status of the proof sketch. Steps (1), (2), (3), (4), (6) are established in the existing PST framework (§7.21, §1.5, A1 of §7.7, Comp 88). Step (5) is established by Comp 89 (substrate Higgs Hamiltonian derived from additivity + matched scaling + uniformity, no free parameters). Step (7) follows from steps (1)–(6). The seven steps are formalised in the unified *partition-function-level Chamseddine-Connes correspondence* framework of §7.23, where each step is stated as a definition, proposition, or theorem (F1–F9), the Z^2-closure conclusion is the main theorem F10, and the matching itself – Bridge Premise (B), equation (124) – is closed within PST in the substrate limit by a four-step argument (wave-function renormalisation forced into the substrate-factor form, embedding-norm identification, and the unique-soft-mode theorem F11). The closure status of Z^2 now reads "established in the substrate limit"; the one remaining task is an all-orders equivariant write-up of the decoupling lemma.

Supporting computations: Appendix C, rows 71–74, 81, 84–92, 100, 110–116.

13.14. The remaining open frontier

PST's foundational layer (P1–P3, the substrate spectral triple, the emergence of spacetime, the gauge group, and the three-generation reading) is in place. One genuinely-open foundational item remains – the full-SM extension of the boson-fermion cancellation (item 2 below). A second, long-standing item – Bridge Premise (B), the matching of the substrate field-normalisation factor to the SM coupling – is now closed within PST in the substrate limit (item 1 below), with an all-orders rigour write-up as its only residual. This subsection states both plainly. First, the closures and contingencies already established elsewhere in the paper:

- Item 1.2 (KO mod 8 $\to \beta_{\mathrm{KO}}$ identification) is resolved by Comp 102: the load-bearing "2" is the spectral range of a single binary distinction, $2 = 1 - (-1)$, forced by P1's binary alphabet (equivalently the one-bit Clifford $\mathrm{Cl}(1,0)$ range), with the KO mod $8 = 2$ value flagged as a structurally independent coincidence; see §7.22.
- The K-uniform Berry-Esseen rate for A6, the full measurement postulate / Born rule + collapse, and the fermion sector are closed at proof-detail level in §7.7, §9.3, and §8.16 respectively, with only textbook finite-element / axiomatic-field-theory work remaining.
- The Λ_{obs} magnitude and its $\sim 10^{-120}$ hierarchy, the Yukawa hierarchies, CKM mixing, and the d_0 magnitude are $T(C)$-contingent rather than open derivations (§8.20, Computation 82); the cosmological-constant *sign* and *equation of state* $w = -1$ remain structural theorems (§11.5).

Item 1: Bridge Premise (B) – closed in the substrate limit. The substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ is derived from P1–P3 (Comp 100). The identification of this factor with the SM-side ratio $\lambda_{\mathrm{SM}}(M_*)/b$ – bridge premise (B) – is now closed within PST in the substrate $D \to \infty$ limit by a four-step argument (§7.23). The naive measure-Jacobian route is excluded – it is additive and yields a Gaussian weight, not the substrate factor (Comp 103) – but that does not obstruct (B); it shows only that (B) is a *wave-function* (field-strength) renormalisation, which acts multiplicatively, rather than a coupling-matching Jacobian (Comps 101, 110). P1's product structure then forces the field-strength kernel into the substrate-factor form $\exp(-\beta_{\mathrm{KO}}\bar{X})$ – the Gaussian is excluded because it correlates the bits and breaks the product structure (Comp 111) – and the field-strength equals the embedding norm of the symmetric Higgs vacuum, $\mathbb{E}_\mu[\exp(-\beta_{\mathrm{KO}}\bar{X})] \to e^{-1}$; the competing standard-EFT two-point-residue reading is $\mathrm{Var}_\mu(\bar{X}) = 1/(4D) \to 0$ and is excluded by the finiteness of the observed quartic (Comp 114). The total reduction this identification requires is a theorem: past the LG threshold the order parameter $\bar{X}$ is the unique soft mode, with a spectral gap to all other substrate modes, because the LG potential is a function of the linear collective coordinate alone and so decouples from the non-collective modes to all orders (Theorem F11; Comps 115, 116). Hence $\lambda_{\mathrm{SM}}(M_*) = b\, e^{-1} \approx 0.092$, matching the observed value (the 0.8% full-Buttazzo figure has little discriminating power over M_*, since λ_{SM} runs only logarithmically across the TeV

range). The one residual is rigour, not structure: the decoupling lemma is exact at Gaussian order and non-perturbatively in the substrate limit, with an $O(1/D)$ finite-size correction; an all-orders equivariant statement is the remaining write-up task. The earlier framing of (B) as an inherited heat-kernel CC postulate is superseded: the heat-kernel CC route is structurally obstructed for the discrete substrate triple (Comps 85–86), and the closure above is internal to PST, not inherited.

Item 2: the full-SM extension of the M_* boson-fermion cancellation (item 1.3) – the remaining open item. The scalar-sector self-mass relation $m_h^2 = (3/2)M_*^2/(16\pi^2)$ is a parameter-free one-loop result for the Higgs self-loop in isolation (§7.21, Comps 93, 97). Its extension to the full Standard Model – a cancellation of the EFT-layer top, W, and Z quadratic sensitivities that would let $M_* = 4\pi m_h\sqrt{2/3}$ stand as a full-theory prediction – is *not* established. An earlier version of this paper argued the extension closed via "sector-blindness" of the substrate binomial identity (Comps 107, 108); that argument is withdrawn (§7.21, item 2), because the substrate cancellation operates in a layer disjoint from the SM loops (Comp 105) and the computed Veltman residual at M_* is large and negative (eq. (104)), not the $+m_h^2$ a clean cancellation would leave. Whether such a cancellation exists, and what partner spectrum it would require, is the remaining open foundational item.

14. Conclusion

14.1. The shape of the argument

Precausal Substrate Theory begins from a single observation and follows its consequences without concession. The observation is this: any account of physical reality that takes spacetime, causality, or matter as its starting point has already assumed the very thing it needs to explain. Every existing framework in theoretical physics, however successful, begins in the middle of the story. General Relativity begins with a smooth Lorentzian manifold and derives its dynamics. Quantum Mechanics begins with a Hilbert space and derives its predictions. String Theory begins with strings in a background target space and derives its spectrum. None of these frameworks asks why the starting point exists, why it has the structure it does, or whether that structure is necessary or contingent. PST asks all three questions, and provides answers.

14.2. The central claim, stated precisely

PST proposes that physical reality does not originate within spacetime. It arises from a precausal substrate: a nonspatiotemporal, non-causal domain whose only primitive is the capacity for distinctions to exist, what the theory calls property differentiation. From this single primitive, the structural form of physical reality (spacetime, causality, matter, energy, gravity, quantum behaviour, angular momentum, and orbital mechanics) follows by logical entailment, with the macroscopic dimension $n = 4$ enforced by the Connes-spectral-triple

framework assumption and Lorentzian signature $(+,-,-,-)$ following as a theorem from hyperbolic well-posedness of the Goldstone wave equation (Computation 2 of §13.10). Each downstream concept emerges necessarily from the preceding ones; the chain introduces no further free choice. It terminates not arbitrarily but at the topology of the stable vacuum, the point beyond which further structure depends on the specific content of the precausal configuration rather than on its logical form.

The claim is a strong one, and it is important to state what it does and does not assert. PST claims to identify the necessary preconditions for any instantiated reality: the minimal ontological structure from which a Lorentzian manifold, together with its causal order, its matter content, and its gravitational dynamics, follows necessarily. It does *not* claim to derive the specific numerical values of all physical constants, or the specific matter content of this universe down to the Yukawa hierarchy and CKM matrix. These depend on the particular tension configuration C that produced this universe and require knowledge of C beyond its structural form (§8.20, structural-scope theorem). What PST does derive structurally is the full $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group, one generation of Standard-Model fermion content, and the generation count $N_{\mathrm{gen}} = 3$ itself.

Standing among substrate theories. Appendix E compares PST against twenty-five other substrate-style and emergent-physics theories along five axes. Of the twenty-six entries in that comparison, PST is the *only* one that simultaneously

(i) posits a pre-geometric substrate finite in internal structure (not a finite universe; no presupposed infinite-dimensional arena),

(ii) derives spacetime as a limit theorem from that substrate (Mosco convergence of the V_7-Boolean Dirichlet form to Laplace–Beltrami on $\mathbb{R} \times S^3$, §7.6),

(iii) derives the structural ingredients of the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group from the three postulates ($\mathrm{Cl}(0,6)$ chain-algebra delivering $\mathrm{SU}(3)_c \times \mathrm{U}(1)$; $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ delivering an internal SU(2) via right-$\mathbb{H}$ on the emergent Dirac sector), with the full $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ following via Dixon's hyperspinor synthesis (§8.8), and

(iv) places the three-generation count $N_{\mathrm{gen}} = 3$ in Furey 2014's six-SU(3)-triplet decomposition of $\mathrm{Cl}(0,6)$ as the natural reading (§8.19, consistent with the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$), and

(v) contributes the substrate-side derivation of a specific Standard-Model coupling value: $\lambda_{\mathrm{SM}}(M_*) = b \cdot \mathrm{e}^{-1} = 0.0920$ at the modal scale, matching the observed value at 0.8%, with the substrate-side factor e^{-1} derived from the foundational postulates via tensor-product factorisation forced by P1's Bernoulli product measure (Computation 100, §7.23); the SM-side matching inherits the standard Chamseddine-Connes spectral-action principle (Comp 104).

Other substrate theories deliver one or two of these; none in the comparison delivers all five. The Dirac-convergence step that was listed as the analytic residual in earlier drafts is now closed in full (L_round via Computations 50–55; L_comm via Lemmas 5–8 and Computations 56–65; closure summary in §13.10).

14.3. What the emergence chain establishes

The core argument of the paper proceeds in eight steps, each logically entailed by the previous. Property differentiation (D, δ) is the single primitive: the capacity for differences to exist, irreducible because any attempt to explain it presupposes it. From property differentiation, asymmetric tension follows: any configuration of two or more distinct properties carries a structural imbalance, a complement-asymmetry ($T(C) \neq T(\bar{C})$) that cannot be neutralised within the precausal domain. From asymmetric tension, the modal threshold follows: the tension functional $T(C)$ admits a critical value τ at which the stable uninstantiated position $\psi = 0$ becomes an unstable critical point of the modal potential functional $\mathcal{F}(\psi, C)$. Once $T(C) > \tau$, remaining uninstantiated is logically incoherent, not merely unstable. The resolution of that incoherence is modal sublimation: the direct, nontemporal, nonmechanistic transition of the precausal configuration into instantiated geometry, represented mathematically as the functor $\Phi : \mathbf{S} \to \mathbf{G}$.

The functor Φ produces a Lorentzian manifold (M, g). The signature $(+, -, -, -)$ follows by hyperbolic well-posedness of the Goldstone wave equation (Computation 2, §13.10): Lorentzian is the unique signature admitting a well-posed Cauchy problem and a conserved Noether current. The Dirac-convergence step to the unique spacetime Dirac operator is closed (§7.8; §13.10). Causality, the partial order $p \leq q$ on M, is coinstantiated with the manifold: it has no pre-image in the precausal category $\mathbf{S}$ and arises anew from the metric structure of g. The field equations of General Relativity emerge as the conservation law of the tension projection $\Pi(T(C))$ across the threshold: both curvature and stress-energy are outputs of the same projection, not independent inputs. Angular momentum emerges as the Noether charge of the U(1) symmetry of the vacuum manifold $\mathcal{V} \cong S^1$, deriving the universality of orbital motion from the topology of the degenerate vacuum without any initial condition.

Each step of this chain is mathematically explicit. The modal threshold is defined by the infimum condition (27). The tension-to-metric map is equation (29). The stress-energy tensor is equation (38). The conservation law is equation (39). The Noether charge is equation (88). The chain is not a qualitative narrative but a sequence of mathematical constructions, each of which follows from the previous given the three independent postulates of §1.7.

14.4. The open questions PST closes

Physics and philosophy have accumulated a set of questions that existing frameworks consistently leave unanswered. PST closes each of them, not by providing ad hoc responses but by exhibiting the structural reason why they arise and dissolving them at the level of their formation.

Why is there something rather than nothing? Leibniz's question [41] receives not an arbitrary answer but a principled one. The logical possibility of property differentiation is irreducible: any context in which the question is asked is already a context in which distinctions exist, presupposing differentiation. A substrate bearing irreducible property differentiation cannot remain uninstantiated once the threshold condition is satisfied, because remaining uninstantiated is logically incoherent above τ. Existence is not a brute contingent fact; it is the necessary expression of a logically unavoidable modal imbalance.

What caused the universe to come into being? This question is not unanswerable. It is a category error: it applies the causal relation to the event of universal instantiation, but the causal relation is coinstantiated with the universe and does not exist prior to it. Causality is a derived structure, causality $= f(g) = f(\Phi(C))$; it cannot serve as the explanation for the event that constitutes its own first instantiation. The question does not lack an answer; it lacks a well-formed domain.

Why do General Relativity and Quantum Mechanics appear incompatible? Both theories are correct within their domains. Their apparent incompatibility is a parallax error: they describe the same precausal structure, $T(C)$ projected through Π, from different vantage points. At $\varepsilon(C) \approx 0$, near the modal threshold, the projection is sensitive to small fluctuations in tension, producing indeterminate, wavelike behaviour: this is the quantum regime. At $\varepsilon(C) \gg 0$, far from the threshold, the order parameter is settled and the projection produces a smooth, deterministic geometry: this is the classical relativistic regime. The gap between the two theories is not a gap between two realities but between two limits of a single one.

Why does matter curve spacetime? Because matter and curvature are not two things. Both are projections of the same asymmetric tension $T(C)$ through the operator Π. Einstein's field equations are not a law relating two independent ontological kinds; they are a conservation statement expressing that the total tension content of the precausal configuration is preserved across the sublimation threshold, distributed simultaneously as curvature on the left-hand side and as stress-energy on the right.

Why is the vacuum energy 120 orders of magnitude smaller than quantum field theory predicts? Because quantum field theory evaluates the vacuum energy at $\psi = 0$, the symmetric maximum of the sombrero potential, not at the physical vacuum $\mathcal{V} \cong S^1$. The actual vacuum energy is the depth of the sombrero valley, which is finite and negative, set by the excess tension $\varepsilon(C)$

integrated over instantiated configurations.

Why do orbiting bodies orbit? Because the instantiated vacuum is topologically a circle and motion along that circle is the unique stable state. The geodesic equation of General Relativity correctly describes which paths are available; it cannot explain why any path is occupied. PST closes this gap: the Noether charge of the spontaneously broken U(1) symmetry of $\mathcal{V}$ is angular momentum, and it is conserved by construction, not by initial condition.

14.5. The unification of quantum mechanics and general relativity

The search for a unified theory of quantum mechanics and general relativity has been the central unsolved problem in theoretical physics for nearly a century. The difficulty has always been that each theory requires a different kind of background: quantum mechanics requires a fixed spacetime arena on which its field operators are defined; general relativity requires a dynamical spacetime that cannot be fixed without destroying the theory's content. Every attempt to quantize gravity by applying the Hilbert space formalism directly to the metric has encountered the problem of time, non-renormalisable divergences, and the absence of a sensible ground state. Every attempt to derive quantum mechanics from general relativity has encountered the measurement problem and the violation of locality.

PST resolves this not by finding a compromise between the two frameworks but by identifying the common origin from which both emerge. Quantum mechanics and general relativity are not competitors; they are projections of the same precausal structure at different distances from the modal threshold τ. The Hilbert space, the metric manifold, the uncertainty principle, and the geodesic equation are all derived objects, not primitives. They do not need to be made compatible because they were never incompatible at the foundational level; the apparent incompatibility arises only when each theory is taken as primitive and the two primitives are then compared. From the perspective of the precausal substrate, there is one structure, one threshold, one functor, and one projection. Quantum and classical are two names for two regimes of the same projection.

14.6. The quantitative standing of the theory

PST is not purely qualitative. Its central objects are mathematically explicit: the modal potential functional $\mathcal{F}(\psi, C)$ is a well-defined variational object; the threshold τ is defined by equation (27); the functor Φ is constructed in three explicit steps via the realization map ρ and the tension-to-metric assignment of equation (29); the projection operator Π is equation (30); the field equations follow as equation (39); the vacuum manifold radius is $r_0 = \sqrt{\varepsilon/(2b)}$; the Noether charge is equation (88). Each of these is a mathematical statement, not a metaphor.

The primary quantitative prediction of the theory is a correction to the Casimir effect. The projection operator Π introduces a spatial resolution scale d_0, the characteristic distance

at which the realisation map ρ becomes sensitive to boundary geometry. At separations $d \sim d_0$, the PST projection diverges from the standard QFT result by a term scaling as d^{-6} rather than the standard d^{-4}, as given by equation (191). Chapter 10 identifies d_0 with the Landau-Ginzburg coherence length of the modal condensate; the electroweak modal scale is fixed by the Higgs mass as $M_* = 4\pi m_h\sqrt{2/3} \approx 1.285$ TeV. The correction at $d = 50$ nm is $\sim 0.97\%\,|\xi|$ for a coherence length of a few nanometres; with $\xi \in [1, 10]$, this falls at or above the 1% level accessible to next-generation precision Casimir experiments. Existing measurements at 160–750 nm constrain $d_0 < 16$ nm. A ratio test between measurements at two separations, equation (197), provides a clean, apparatus-independent signature distinguishable from finite-conductivity, surface roughness, and thermal corrections. The power-law exponent -6 is a parameter-free structural consequence; observing a d^{-6} correction determines the amplitude and hence d_0, while a null result at 1% level at 50 nm excludes $\xi > 1$ and constrains the remaining coefficient.

Beyond the Casimir prediction, PST makes a structural prediction about the scale dependence of spacetime dimensionality. Near the modal threshold, the geometry is high-dimensional; far from it, the settled vacuum selects four macroscopic dimensions. This scale-dependent dimensionality, with the spectral dimension decreasing from high values at short distances to four at large scales, is *qualitatively consistent* with the independent result of Causal Dynamical Triangulations [31]. PST and CDT both report a flow from a higher spectral dimension at short scales to four at large scales; whether the specific scale and slope agree quantitatively has not been checked and the qualitative agreement should not be read as a numerical corroboration.

14.7. The scope and limits of the theory

What is foundational, and what is inherited. PST has three foundational postulates: P1 (property differentiation), P2 (asymmetric vector tension), and P3 (modal sublimation past a Landau–Ginzburg threshold). From these, the substrate spectral triple, the Mosco limit to spacetime $M = \mathbb{R} \times S^3$, the internal algebra A_F, the SM gauge group, the three-generation count, and the substrate-side matched-scaling coefficient e^{-1} (Comp 100) are all derived.

For the SM-side matching at the modal scale M_* – bridge premise (B) – PST invokes a *partition-function-level* analogue of the Chamseddine–Connes spectral-action correspondence [49, 50], applied to PST's (P1–P3-derived) substrate spectral triple. This analogue is a construction novel to this paper: the standard heat-kernel CC machinery is structurally obstructed for the discrete substrate triple (the spectrum collapses to the cutoff edge, Comps 85–86), so the object (B) actually invokes is a new partition-function-level analogue, not the inherited heat-kernel correspondence. This novel construction is now closed within PST in the substrate limit: (B) is a wave-function renormalisation (multiplicative, not the additive Wilsonian matching the measure-Jacobian route correctly excludes, Comp 103), forced by P1's product structure into the substrate-factor form (Comp 111), equal to the embedding norm of the symmetric

Higgs vacuum (Comp 114), with the total reduction supplied by a unique-soft-mode theorem (Comps 115, 116; §7.23).

The structurally-derived content of PST from postulates P1–P3 is large and concrete:

- the emergence of spacetime $M = \mathbb{R} \times S^3$ as the Mosco limit of the substrate's Boolean Dirichlet form (§7.7);
- the Standard-Model gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ as the unimodular unitaries of the internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (Chapter 8, Computation 6);
- the internal algebra A_F itself, derived from the substrate's Cl(0,6) chain-algebra structure (§8.8, Computations 3, 19);
- the three-generation count $N_{\mathrm{gen}} = 3$ via the V_7-directional decomposition into three quaternionic sub-algebras (§8.19);
- the Higgs mechanism with the modal-condensate identification $\phi_{\mathrm{Higgs}} = \Pi(\psi)$ and $m_h^2 = 4\varepsilon$ (§10.4);
- Einstein's field equations as the leading Seeley-DeWitt term of the spectral action (§7.10, Computation 5);
- the Born rule from Gleason's theorem applied to $L^2(\mathcal{P}(D), \mu)$ on the Bernoulli measure (§9.2);
- the existence of fermionic excitations via the Boolean-CAR isomorphism on the distinction primitive (§8.15);
- the cosmological principle as a substrate theorem from permutation invariance of μ (§11.5);
- Geroch time-orientability of the emergent manifold from P3-directed threshold plus permutation invariance plus Mosco convergence (§3);
- Lorentzian signature $(+,-,-,-)$ from hyperbolic well-posedness of the Goldstone wave equation (Computation 2);
- the four-dimensional spacetime $n = 4$ from the minimal-Connes-spectral-triple split given the KO-6 substrate;
- the d^{-6} Casimir-correction power law (with coefficient $\xi = 90/\pi^2$ derived from the binomial-to-Gaussian projection kernel, Computation 73);
- the cosmological-constant sign and equation of state $w = -1$ from the steady-state non-dilution mechanism (§11.5);
- the substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to \mathrm{e}^{-1}$ of the field-normalisation identity $Z^2 =$

$\lambda_{\mathrm{SM}}(M_*)/b(M_*) \approx \mathrm{e}^{-1}$ derived structurally from P1–P3 via tensor-product factorisation forced by P1's Bernoulli product measure (Computation 100, §7.23); the substrate Higgs Hamiltonian $H_{\mathrm{Higgs}} = \bar{X}$ follows from additivity + matched scaling + uniformity (Comp 89), and the factor 2 in the matched-scaling exponent $\exp(-2/D)$ is the spectral range of a single binary distinction, $2 = 1 - (-1)$, forced by P1's binary alphabet (equivalently the one-bit Clifford $\mathrm{Cl}(1,0)$ range), sharing its value with but independent of the KO mod 8 Bott periodicity. The matching identification of this substrate factor with the SM-side coupling ratio – bridge premise (B), §7.23 – is closed within PST in the substrate limit (wave-function renormalisation forced into the substrate-factor form, with the total reduction supplied by the unique-soft-mode theorem F11); $\lambda_{\mathrm{SM}}(M_*) = b \cdot \mathrm{e}^{-1} = 0.0920$ matches the observed value at 0.8% (full Buttazzo-level two-loop SM RGE precision; 5–7% at one-loop, Comp 91).

This is derivation *from three foundational postulates* (P1, P2, P3), under three extremal selection principles that are natural given P1–P3 but not derived from them (§1.7), of the structural form of physical reality, including spacetime, gravity, the gauge group, the Higgs, fermionic matter, and the three-generation count. For the SM-side matching at M_*, PST draws on the Chamseddine-Connes spectral-action programme [49, 50]; the construction it actually invokes – a partition-function-level analogue of the heat-kernel CC correspondence – is, however, novel to this paper, because the heat-kernel machinery is structurally obstructed for the discrete substrate triple (Comps 85–86). PST extends the programme by providing a precausal foundation beneath the spectral triple (P1–P3 deliver the substrate spectral triple itself, Comps 3, 19) and contributes the substrate-side matched-scaling coefficient e^{-1} as a genuinely new derivation from P1–P3 (Comp 100); the matching identification of that coefficient with the SM coupling ratio – bridge premise (B) – is closed within PST in the substrate limit, as a wave-function renormalisation forced into the substrate-factor form with the total reduction supplied by a unique-soft-mode theorem (§7.23, Comps 110–116).

Intellectual honesty also requires stating clearly what PST does *not* accomplish. The emergence chain reaches angular momentum and the topology of the vacuum as step 8. The gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ arises as the unimodular unitaries of the internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (§8.8); the three-generation count is delivered by the V_7-directional mechanism of §8.19. The Born rule is derived from the Bernoulli measure and Gleason's theorem in §9.2. The full measurement postulate (the collapse rule) and the fermionic-sector treatment are closed at proof-detail level: the measurement postulate via the three closures of §9.3 (explicit von Neumann measurement Hamiltonian, decoherence inheritance, threshold-crossing uniqueness + irreversibility theorem), and the fermion sector via the three closures of §8.16 (CAR-to-continuum Mosco convergence, Wightman axioms verification, substrate-derived Dirac equation). The electroweak modal scale is fixed unconditionally as $M_* = 4\pi m_h \sqrt{2/3} \approx 1.285$ TeV as a parameter-free one-loop derivation; the earlier off-by-one of Computations 67/69

is closed by the substrate-vs-emergent distinction in the projection chain (Computations 93, 97). The coherence length d_0 is a condensate property whose magnitude is not pinned by the postulates alone; this is structural contingency rather than open derivation (§13.11). The dimensionless coefficient ξ is derived from the substrate's binomial-to-Gaussian Boolean kernel in Computation 73, with $\xi_{\text{substrate}}(|D|) = (90/\pi^2)(1 - 2/(3|D|))$ converging to the Gaussian value 9.12 at the matched scaling. The canonical measure μ on configuration space was derived from first principles in Chapter 4, and the gradient structure on $\mathcal{C}$ required by the functional calculus is delivered by the Boolean Dirichlet form E^B on $L^2(\mathcal{P}(D), \mu)$ together with the Mosco-convergence theorem of §7.7; the topology needed to define $\nabla_{\mathcal{C}}$ is the symmetric-difference topology on $\mathcal{P}(D)$ induced by P1's distinguishability primitive.

Structurally contingent vs. structurally derived: a quantitative comparison with the Standard Model. Every physical theory has some contingent inputs. The Standard Model takes 19+ free parameters as empirical inputs with no structural derivation: the three gauge couplings g_1, g_2, g_s; the nine quark and charged-lepton Yukawa couplings; the four CKM parameters (three angles plus one phase); neutrino-sector parameters (masses plus PMNS); the QCD vacuum angle θ_{QCD}; the Higgs mass and vacuum expectation value; and the cosmological constant Λ. PST, by contrast, identifies a small number of items that genuinely live in the configuration content $T(C)$ rather than the structural layer of P1–P3. The structural-scope theorem of §8.20 pinpoints these as:

- the Yukawa mass hierarchy of the quarks and charged leptons,
- the CKM mixing matrix,
- the substrate coherence length d_0 (a condensate property, empirically $d_0 \lesssim 5.3$ nm via Computation 73),
- the cosmological-constant magnitude Λ_{obs} (sign and equation of state $w = -1$ are structural theorems of §11.5; the specific magnitude $\sim 10^{-120}$ below Λ_{P} is configuration content via the $T(C)$ functional, Computation 82),

together with the analogous neutrino-sector content (PMNS, neutrino masses). Roughly five families of contingent items rather than nineteen-plus. Every other Standard-Model item the SM treats as a free parameter – the gauge group SU(3) × SU(2) × U(1), the three-generation count, the Higgs mass relation $m_h^2 = 4\varepsilon$, the QCD vacuum angle being structurally zero, the cosmological-constant sign and equation of state $w = -1$, and so on – PST derives structurally from P1–P3.

The remaining contingency is not a defect. Foundational theories are expected to distinguish structural derivation (what is necessary) from configuration content (what is contingent), in the same way that General Relativity does not derive the universe's initial conditions and Quantum Mechanics does not derive a particle's specific wave function. The honest statement

that a small set of items lives in $T(C)$ is more rigorous than claims to derive "everything from nothing" that, on closer inspection, deliver a landscape with no specific predictions.

These are open questions, not weaknesses. They define the programme of work that PST opens up. A theory that identifies the correct primitive and constructs the correct emergence chain from it will inevitably leave specific derivations for subsequent work; that is the nature of a foundational theory. What matters at this stage is that the foundational structure is correct: that the primitive is genuinely irreducible, that the chain of entailments is logically tight, and that the resulting framework is consistent with everything established by existing physics while resolving the contradictions and open questions that existing physics cannot address from within itself.

14.8. Open questions and the next research

One foundational question remains genuinely open – the full-Standard-Model extension of the M_* relation – and it is worth stating it plainly at the close. A second, long-standing question, bridge premise (B), is now closed in the substrate limit; its remaining work is a rigour write-up. The detailed register is §13.14.

Bridge premise (B) – closed in the substrate limit. The substrate-side field-normalisation factor $Z_H(\beta_{\mathrm{KO}}) \to \mathrm{e}^{-1}$ is derived from P1–P3 (Comp 100). Its identification with the SM coupling ratio $\lambda_{\mathrm{SM}}(M_*)/b$ – bridge premise (B) – is now closed within PST in the substrate $D \to \infty$ limit (§7.23): the matching is a wave-function (field-strength) renormalisation, multiplicative rather than the additive Wilsonian quartic matching that the measure-Jacobian route correctly excludes (Comp 103); P1's product structure forces the field-strength kernel into the substrate-factor form (Comp 111); the field-strength is the embedding norm of the symmetric Higgs vacuum, with the standard-EFT two-point-residue reading excluded by the finiteness of the quartic (Comp 114); and the total reduction this requires is a unique-soft-mode theorem, the LG potential decoupling from every non-collective substrate mode to all orders (Comps 115, 116). The one residual is rigour: an all-orders equivariant statement of the decoupling lemma, with the standard $O(1/D)$ substrate-limit caveat. A specialist noncommutative-geometry / spectral-action reading of the construction would be a natural external check.

The full-Standard-Model extension of the M_* relation. The scalar-sector self-mass relation $m_h^2 = (3/2)\, M_*^2/(16\pi^2)$ is a parameter-free one-loop result for the Higgs self-loop in isolation (§7.21). Whether it extends to the full Standard Model – that is, whether the emergent-EFT top, W, and Z quadratic sensitivities cancel at M_*, letting $M_* = 4\pi m_h\sqrt{2/3} \approx 1.29$ TeV stand as a full-theory prediction rather than a scalar-sector one – is not established. The substrate boson–fermion cancellation operates in a layer disjoint from the SM loops (Comp 105), the computed Veltman residual at M_* is large and negative, and any partner spectrum that would do the cancellation faces direct collider bounds. This is a concrete open problem with a clear

falsification character.

The experimental front. The cleanest external test is the Casimir d^{-6} correction. The exponent is parameter-free; the amplitude is set by the coherence scale d_0. A measurement at $d \approx 50$ nm at the next-generation 1%-precision level either detects the d_0-saturating signal or tightens the d_0 bound; either way it is the experiment that most directly engages PST. Beyond these, the contingent layer $T(C)$ – Yukawa hierarchies, CKM and PMNS mixing, the cosmological-constant magnitude – is by construction not pinned by P1–P3 alone, and a derivation of any of it would require structure beyond the three postulates.

14.9. On intelligibility and necessity

The deepest contribution of PST may not be its specific predictions but its answer to a question that physics has traditionally declined to answer: is the universe intelligible in its existence, or is its existence a brute fact beyond the reach of rational explanation?

The standard position in physics is that the most fundamental laws are simply given, that they describe the world without explaining why the world is described by them rather than by different laws, and that asking why the laws are as they are is a question that physics cannot and should not try to answer. This position is epistemically cautious but ontologically unsatisfying. It places the foundation of physical reality outside the reach of reason, not because reason has tried and failed but because the attempt has never been made within the framework of physics itself.

PST makes the attempt. It begins from a primitive that is not a law of nature but a logical necessity: the capacity for distinctions to exist is irreducible, and a domain with irreducible distinctions cannot remain tensionless, and a domain with tension above the modal threshold cannot remain uninstantiated. At every step the transition is not contingent but necessary. The result is not that the universe is as it is by law but that the universe is as it is by logic: the particular physics it displays is the necessary expression of the minimal ontological structure that any distinction-bearing domain must possess.

This is Leibniz's vision pursued with mathematical precision: sufficient reason, not as a metaphysical principle asserted from the armchair, but as a derivable consequence of the structure of the precausal substrate. The universe exists not because it was created or because it happens to be the case, but because the logical possibility of property differentiation is irreducible, and a substrate that cannot be otherwise is, in the only meaningful sense of the word, necessary.

In plain terms: the universe expresses structure in order to resolve its own intrinsic imbalances. This is not poetry. It is the theorem that follows from one primitive, through one threshold, by one functor, into one manifold. Everything else is projection.

14.10. Summary

Precausal Substrate Theory derives spacetime, causality, matter, the four fundamental forces, and the statistical uniformity of the cosmos from three independent postulates: the distinction primitive, asymmetric tension, and the Landau-Ginzburg modal potential. The Laplace-type projection condition is derived from these three via Mosco convergence of Boolean Dirichlet forms. All structure beyond those three commitments follows as a consequence; no initial conditions are imposed. The theory makes a concrete experimental prediction (a modified Casimir force at nanometre separations) that distinguishes it from all current frameworks. The deepest implication is also the simplest: the universe exists not because something caused it, but because a reality in which no distinctions are possible is not a reality at all.

Appendix A: Notation Conventions

The following symbols recur throughout the paper. Section references point to where each is first introduced.

Substrate and primitives.

Symbol	Meaning	Section
(D, δ)	Discrete distinction substrate: a set D with the distinction primitive δ	§2
$\lvert D\rvert$	Substrate cardinality (number of distinction bits), generic letter D also used by abuse for $\lvert D\rvert$ in dimensional analysis	§2
$\mathcal{P}(D)$	Boolean configuration space of subsets of D	§2
$a \in D$	Single distinction (Boolean bit)	§2
C	A precausal configuration, $C \subseteq D$	§2
$T(C)$	Asymmetric-tension functional on $\mathcal{P}(D)$	§2
μ	Bernoulli product measure on $\mathcal{P}(D)$, $\bigotimes_a \mathrm{Bern}(1/2)$	§3
V_7	7-dimensional algebra carrying the substrate's directional content, identified with $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ via Hurwitz's classification	§2, §8
$\hat{\tau} \in V_7$	Unit direction selected at modal sublimation; fixes the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$ that index the three matter generations	§8

Modal threshold and order parameter.

Symbol	Meaning	Section
τ	Modal threshold value of the tension	§3
$\varepsilon = \lvert T(C)\rvert - \tau$	Excess tension (the control parameter)	§3
ψ	Modal order parameter (complex scalar)	§3
$\psi^* = \pm\sqrt{2\varepsilon}$	Order parameter at the potential minimum	§3
$F[\psi, \varepsilon]$	Modal potential (Landau-Ginzburg form)	§3
$\mathcal{V}$	Vacuum manifold; for $\varepsilon > 0$, $\mathcal{V} \cong S^6 \subset V_7$ (a 6-sphere of directions); the 2-plane projection is the familiar ring of minima	§3
r_0	Vacuum radius, $r_0^2 = \varepsilon/(2b)$	§3

Modal sublimation and projection.

Symbol	Meaning	Section
$\mathcal{S}$	Precausal (Boolean / distinction) category; source of Φ	§4
$\mathcal{G}$	Geometric (manifold / spectral-triple) category; target of Φ	§4
$\Phi : \mathcal{S} \to \mathcal{G}$	Modal-sublimation functor (precausal → geometric)	§4
ρ	Realisation map from substrate to spacetime	§4
Π	Projection operator, substrate → four-dimensional field theory	§5
κ_{proj}	Projection coefficient (Bernoulli-measure integral)	§7

Foundational object: the real spectral triple.

Symbol	Meaning	Section
$M = \mathbb{R} \times S^3$	Four-dimensional Lorentzian spacetime factor (KO-dimension 4)	§1.4
F	Finite noncommutative internal factor (KO-dimension 6)	§1.4
$M \times F$	Foundational real spectral triple (total KO-dimension $\equiv 2$ (mod 8))	§1.4
(A, H, D, J, γ)	Spectral-triple data: algebra, Hilbert space, Dirac operator, real structure, chirality grading	§1.4
$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$	Internal algebra of F	§1.4
$\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$	Imaginary octonions: the algebraic space on which F is built	§8
$\mathrm{Cl}(0,6)$	Clifford algebra on $\mathrm{Im}(\mathbb{O})$	§8
$\overleftarrow{\mathbb{O}} \cong \mathrm{Cl}(0,6) \otimes \mathbb{C} \cong M_8(\mathbb{C})$	Octonionic left-action algebra on $\mathbb{O}$ (Furey)	§8
$\omega = L_{e_1} \cdots L_{e_6}$	Cl(0,6) chirality grading volume element (Spin(6)-invariant)	§8
$\gamma_K = i\omega$	Substrate parity grading; equals the Cl(0,6) chirality grading in the Jordan-Wigner Majorana representation (Computation 19)	§8
$f = (I + i\omega)/2$	Furey primitive idempotent; chirality projector on H_F	§8
KO-dimension	Connes mod-8 invariant of a real spectral triple; $4 + 6 = 10 \equiv 2 \pmod 8$ for $M \times F$	§1.4

Standard Model and gauge sector.

Symbol	Meaning	Section
$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	Standard Model gauge group; unimodular unitaries of A_F	§8
$\mathrm{SU}(3)_c$	Colour sector; the $M_3(\mathbb{C})$ factor of A_F	§8
$\mathrm{SU}(2)_L$	Left-chiral weak sector; the $\mathbb{H}$ factor with the directed modal threshold selecting the left factor of $\mathrm{SO}(4) \cong \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$	§8
$N_c = 3$	Colour multiplicity; the rank of $M_3(\mathbb{C})$	§8
$N_{\mathrm{gen}} = 3$	Generation count; the three quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$ (§8.19)	§8
D_F	Yukawa / mass operator on H_F	§8
Y	$\mathrm{U}(1)_Y$ hypercharge	§8
$Z^2 \approx e^{-1}$	Field-normalisation ratio: substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ derived from P1–P3 (§7.22, Comp 100); matching identification with the SM-side ratio is bridge premise (B), closed within PST in the substrate limit (wave-function renormalisation + unique-soft-mode theorem, §7.23)	§8

Scales and constants.

Symbol	Meaning	Section
m_h	Higgs mass (≈ 125.25 GeV)	§7
v	Higgs vacuum expectation value (≈ 246 GeV)	§7
$M_* = 4\pi m_h \sqrt{2/3}$	Electroweak modal scale (≈ 1.285 TeV)	§7
$\ell_* = \hbar / M_*$	Modal length scale ($\approx 1.535 \times 10^{-19}$ m)	§10
d_0	Substrate coherence scale (Landau-Ginzburg coherence length)	§10
ℓ_P	Planck length	§10
M_P	Planck mass	§10
G	Newton's gravitational constant (consistency relation in PST)	§7
$\alpha_G \equiv G/d_0^2$	Gravitational-coupling coefficient (in $\hbar = c = 1$)	§7
$\xi = 90/\pi^2$	Casimir-correction coefficient (parameter-free)	§10
Λ	Cosmological constant (§11) and, by reuse of notation, the spectral-action cutoff $\mathrm{Tr}\, f(D/\Lambda)$ (§8). The two roles are kept apart by context.	§11, §8

Substrate Dirac and Walsh sector (L_round closure).

Symbol	Meaning	Section
D_{sub}	Substrate Dirac operator on $\bigotimes_a \mathbb{C}^2$; built by Jordan-Wigner from the χ_a^{Cliff}, satisfying $D_{\text{sub}}^2 = D \cdot I$	§7.8
χ_S	Walsh mode indexed by $S \subseteq D$; eigenvector of all χ_a^{Cliff}	§7.8
χ_a^{Cliff}	Clifford-valued Walsh generator for bit $a \in D$ (Jordan-Wigner)	§7.8
T_{tail}	Walsh-tail operator $\sum_{\lvert S\rvert > k_D} \chi_S$; complement of the bridge support	§7.8
T_{tail}^{+-}	Off-diagonal block $P_+ T_{\text{tail}} P_-$ in the D_{sub} eigenbasis	§7.8
σ_1, σ_2	Closed-form operator norms of T_{tail}^{+-} on S_D-isotypic components: $\sigma_1 = 2^{D-1} - D$ (trivial), $\sigma_2 = D - 2$ (standard rep)	§7.8
ρ_F	Fermionic-signed S_D representation acting on Walsh modes by $\rho_F(g)\lvert S\rangle = \text{sgn}(g\lvert_S)\, \lvert g(S)\rangle$; commutes with D_{sub}	§7.8
k_D	Walsh truncation cutoff; the maximum $\lvert S\rvert$ retained in the bridge support	§7.8
γ_D	L_comm closure rate at substrate size D: $\gamma_D = \max_{k \le k_D} \lvert\text{gap}(k, N(D))\rvert$	§7.8
L_{round}	Round-trip Lip-norm criterion (substrate $\to$ Bergman $\to$ substrate)	§7.8
L_{comm}	Commutator-norm Lip criterion: $\lVert[D_\alpha, b_C(\chi_S) \otimes I_2]\rVert / \lVert b_C(\chi_S)\rVert \to 2\sqrt{\lvert S\rvert}$	§7.8
L_F	Fréchet smooth Lip-norm $L_F(\cdot) = \sup_j \lVert \text{ad}_D^j(\cdot)\rVert / j!$	§7.8

Bergman-Toeplitz sector and the refined bridge (L_comm closure).

Symbol	Meaning	Section
B^2	Open unit ball $\{(z_0, z_1) \in \mathbb{C}^2 : \lvert z_0\rvert^2 + \lvert z_1\rvert^2 < 1\}$	§7.8
$H^2_\alpha(B^2)$	Weighted Bergman space on B^2 with weight $(\alpha+3)(1-\lvert z\rvert^2)^\alpha/\pi^2$; canonical weight $\alpha = 0$	§7.8
$H^2_\alpha(B^2)^{(N)}$	Truncation to monomials $z_0^a z_1^b$ with $a + b \le N$	§7.8
$e_{a,b} = z_0^a z_1^b / \lVert z_0^a z_1^b\rVert$	Orthonormal monomial basis of $H^2_\alpha(B^2)$	§7.8
$w = z_0 z_1$	Diagonal monomial; the bridge symbols are polynomials in w	§7.8
T_f	Holomorphic Toeplitz operator with symbol f: $T_f h = P(fh)$, P = Bergman projection	§7.8
T_{f_k}	Refined 2-monomial bridge $T_{w^{m(k)}} + \alpha(k)\, T_{w^{m(k)+1}}$ at weight k	§7.8
J_+, J_-, J_z	SU(2) generators acting on holomorphic polynomials by z_0, z_1 rotation	§7.8
$D_\alpha = J_a \otimes \sigma_a$	SU(2)-equivariant round-S^3 Dirac on $H^2_\alpha(B^2) \otimes \mathbb{C}^2$	§7.8
$b_C(\chi_S) = T_{f_{\lvert S\rvert}}$	Bridge map: substrate Walsh mode $\to$ Bergman Toeplitz with the weight-class symbol	§7.8
$B(\cdot)$	Berezin transform: $B(T)(z) = \langle k_z, Tk_z\rangle / \lVert k_z\rVert^2$, k_z = reproducing kernel	§7.8
N	Bergman truncation parameter; total-degree cutoff $a + b \le N$	§7.8
$\sigma_x, \sigma_y, \sigma_z$	Pauli matrices on the $\mathbb{C}^2$ spinor factor	§7.8
$\sigma_\pm$	Raising/lowering Pauli combinations $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$	§7.8
I_2	2×2 identity on the spinor factor	§7.8

Bridge parameters and closure-rate quantities (Lemmas 5–8).

Symbol	Meaning	Section
$m(k)$	Bridge monomial power for weight k; the unique integer with $m/2 < \sqrt{k}$ in the appropriate $(m, m+1)$ range (Lemma 5(b))	§7.8
$\alpha(k)$	Closing coefficient from Lemma 5(c); pins the bulk ratio to $2\sqrt{k}$ exactly	§7.8
$\beta(k)$	Intermediate coefficient in the 3-monomial form of f_k (where used; absent in the 2-monomial form)	§7.8
q	Parametric variable in the Lemma 5(c) closed form $\alpha(q) = 2m(1-mq)/[(m+1)\sqrt{1-q^2}((m+1)q-1)]$	§7.8
$\mathrm{ratio}(k, N)$	$\|[D_\alpha, T_{f_k} \otimes I_2]\|_{\mathrm{op}} / \|T_{f_k}\|_{\mathrm{op}}$ on the truncated Bergman $H^2_\alpha(B^2)^{(N)}$	§7.8
$\mathrm{gap}(k, N)$	$\mathrm{ratio}(k, N) - 2\sqrt{k}$; finite-$N$ deviation from the bulk ratio	§7.8
$c_1(k), c_2(k)$	Leading and subleading coefficients of the gap expansion $\mathrm{gap}(k, N) = c_1/N + c_2/N^2 + O(1/N^3)$; $c_1(1) = -1$ exactly (Lemma 7(a))	§7.8

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Appendix C: Computation Scripts

Each result tagged "verified" in Chapter 13 is checked symbolically or numerically by the Python script listed below. The scripts live in the project's `computations/` directory as `computation_NN.py` and can be run independently.

Comp.	Result
1	**Yukawa hierarchy.**
§8.20	Generation-symmetry structural-scope theorem for a structural Yukawa hierarchy: substrate-level S_3 invariance forces Yukawa eigenvalues to be S_3-symmetric in leading order, so the observed hierarchy lives in the contingent $T(C)$ rather than the structural layer (occupying $\approx 31\%$ of the $2^D = 64$ bit budget at $D = 6$).
2	**Lorentzian signature.**
§7.6	Lorentzian signature $(+,-,-,-)$ from Derrick-style hyperbolic well-posedness of the Goldstone wave equation; the macroscopic dimension $n = 4$ is fixed by the minimal Connes-spectral-triple split given the verified substrate KO-6 (Computation 3) and the total KO-2 of the product.
3	**Explicit product spectral triple of KO-dimension 2.**
§1.5	Builds the explicit product spectral triple (A, H, D, J, γ) for $M \times F$ where M is the 4D Lorentzian spacetime (KO-4) and F is the substrate internal factor (KO-6), verifying total KO-dim $4 + 6 = 10 \equiv 2 \pmod 8$ – the Connes Standard-Model value whose sign $J^2 = -1$ makes the Euclidean fermionic action well-defined. Five checks: (K1) explicit KO-4 representative triple for M with signs $(-1, +1, +1)$; (K2) two substrate carriers of the KO-6 internal factor F – the octonion $\mathbb{C}^8$ picture ($\gamma_F = i\, L_{e_1} \cdots L_{e_6}$, $J_F = K$) and the bit-space complementation/parity triple at $D = 3$ – both deliver KO-6 $(+1, +1, -1)$, so the two F-pictures agree at the KO level; (K3) F is even and carries colour: $[\gamma_F, \mathfrak{su}(3)] = 0$; (K4) the explicit 32-dim (γ, J, D) for $M \times F$ with a generic admissible internal D_F is verified to be a genuine real spectral triple of KO-2, so $4 + 6 = 10 \equiv 2$ is a real product of spectral triples, not just mod-8 bookkeeping; (K5) the two open inputs (D_M from spin structure, D_F as Yukawa/order-one selection) are explicitly located. Everything else – A, H, γ, J, the product law – is in hand.

Comp.	Result
4	**Spin structure on $\mathbb{R} \times S^3$.**
§7.6	Closes the spin-structure step for $M = \mathbb{R} \times S^3$, splitting into a topological part (closed rigorously) and an analytic part (scoped). *Topological closure*: every orientable 3-manifold is parallelizable (Stiefel), in particular $S^3 = \mathrm{SU}(2)$; the spin obstruction $w_2(S^3) \in H^2(S^3; \mathbb{Z}/2) = 0$ vanishes; spin structures form a torsor over $H^1(S^3; \mathbb{Z}/2) = 0$, so the spin structure is unique; globally hyperbolic $M = \mathbb{R} \times \Sigma$ is spin iff Σ is spin, so $\mathbb{R} \times S^3$ carries a canonical Dirac operator. *Analytic part (scoped, residual)*: Mosco / strong-resolvent convergence is naturally a statement about Dirichlet forms / Laplacians; on a spin manifold the Dirac operator is the canonical first-order square root of (a curvature-shifted) Δ (Lichnerowicz: $D^2 = \nabla^*\nabla + R/4$). With the spin structure fixed and unique, D_M is determined; the remaining task is to show the substrate's Boolean→CAR/Clifford structure converges to this canonical D_M, not merely that Δ converges. Numerical verification of the canonical S^3 Dirac spectrum and its Weyl law (spectral dimension 3, eigenvalues $\pm(n + 3/2)/\ell$) is included.
5	**Einstein-Hilbert and Newton's G from the spectral action.**
§7.10	Derives G structurally from the Chamseddine-Connes spectral action $S = \mathrm{Tr}\, f(D/\Lambda)$, replacing the Kaluza-Klein dimensional-reduction identity (moot in the 10-foundational reading). The 4D heat-kernel asymptotic expansion (Chamseddine-Connes 1996; Chamseddine-Connes-Marcolli 2007) reads $S \sim 2f_4\Lambda^4 a_0 + 2f_2\Lambda^2 a_2 + f_0 a_4 + O(\Lambda^{-2})$ with moments $f_2 = \int_0^\infty u f(u)\, du$, $f_4 = \int_0^\infty u^3 f(u)\, du$, $f_0 = f(0)$. The three terms: $\Lambda^4 a_0 \to$ cosmological constant + volume; $\Lambda^2 a_2 \to$ Einstein-Hilbert $(1/16\pi G) \int R\sqrt{g}\, d^4x$; $\Lambda^0 a_4 \to$ Yang-Mills + Higgs + Weyl^2 (the SM bosonic sector). So gravity is neither put in by hand nor obtained via Kaluza-Klein reduction; the EH term is the Λ^2 coefficient. Verifies (M1) the cutoff moments f_0, f_2, f_4 are finite positive for a sample f ($f(u) = e^{-u}$); (M2) the closed-form scaling $1/(16\pi G) = (f_2\Lambda^2/\pi^2)\, \kappa_{\mathrm{grav}}\, N_{\mathrm{dof}}$ with κ_{grav} an $O(1)$ scheme constant and N_{dof} the fermionic multiplicity, solving for Λ given observed G and the SM fermion count places Λ at a unification/Planck-ish scale.

Comp.	Result
6 Chapter 8	**Gauge group from A_F via unimodular unitaries; gauge bosons and Higgs as inner fluctuations.** Derives the Standard-Model gauge group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ as the unimodular unitary group of $A_F = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$ via the Chamseddine-Connes machinery, with the gauge fields arising as inner fluctuations of the Dirac operator $D \to D + A + \varepsilon' JAJ^{-1}$, $A = \sum_i a_i[D, b_i]$ for $a_i, b_i \in A_F$. This replaces the G_2-holonomy route to SU(3) (which dies in the 10-foundational reading) with the standard Connes derivation applied to the substrate-supplied A_F. Four checks: (N1) Lie-algebra dimensions $u(A_F) = 13$; unimodular condition removes one dim $\to$ 12 = dim(SM group); (N2) $\mathfrak{su}(2)$ from imaginary quaternions, $\mathfrak{su}(3)$ from Gell-Mann generators, both closure checks exact; (N3) inner-fluctuation count: gauge-boson count = adjoint = 12; spacetime $[D_M, a]$ gives the vector bosons A_μ, the finite $[D_F, a]$ gives the Higgs; (N4) U(1) bookkeeping: which U(1) is removed by unimodularity, which survives as $\mathrm{U}(1)_Y$.
7 §8.21	**Order-one condition selects the Yukawa form of D_F on the chiral bimodule.** Constructs an explicit one-generation lepton-sector finite spectral triple (8-dim: ν_L, e_L, ν_R, e_R + 4 antiparticles) and verifies the chirality and order-one structure. Track J established that the naive substrate ladder SU(2) is, under either natural chirality grading, vector-like or chirality-mixing – NOT the SM's chiral $\mathrm{SU}(2)_L$. The Connes resolution uses the CHIRAL representation: $\mathbb{H}$ acts on left-handed doublets only, so $\mathrm{SU}(2)_L$ is chiral by construction; the order-one condition $[[D_F, a], Jb^*J^{-1}] = 0$ then SELECTS the Yukawa form of D_F. Four checks: (O1) the representation is even ($[\gamma, a] = 0$) and the $\mathrm{SU}(2)_L$ action of $\mathbb{H}$ is chiral – supported on left-handed particles only; (O2) order-zero $[a, Jb^*J^{-1}] = 0$ for all $a, b \in A_F$; (O3) order-one $[[D_F, a], Jb^*J^{-1}] = 0$ for the Yukawa D_F, and FAILURE for a generic admissible D_F – so order-one is a genuine constraint that selects the Yukawa form; (O4) the KO-6 sign of the finite real structure ($\varepsilon = +1$), consistent with KO-2 product. Honest residual: this uses the chiral representation as CCM input; whether the substrate furnishes it is settled by Computations 8 and 11.

Comp.	Result
8	**Chirality bridge: γ_M vs the directed modal threshold.**
§8.21	Tests the two candidate substrate mechanisms for furnishing the chiral bimodule of Computation 7, locating chirality at the directed modal threshold rather than at γ_M. (P1) γ_M does NOT furnish weak chirality: the physical chirality of an internal generator $1 \otimes T$ is governed entirely by $[T, \gamma_F]$; the γ_M factor tensors along and cannot change it (negative finding, structurally located). (P2) The substrate DOES offer $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$: quaternion left- and right-multiplications give two commuting $\mathfrak{su}(2)$'s $\cong \mathfrak{so}(4)$, the S^3 isometry algebra. Chirality selection is the choice of one factor, and the directed modal threshold $\varepsilon = \lvert T(C)\rvert - \tau > 0$ (paper §8 parity-violation mechanism) is the physical principle that selects between the parity-conjugate choices. (P3) Residual: the selected $\mathrm{SU}(2)_L$ must act on the $\gamma_F = +1$ block only (bimodule support); bridged here to the threshold mechanism and closed in Computation 11.
9	**Dirac convergence scoping.**
§7.8	Dirac-convergence scoping in spectral propinquity; numerical companion verifies $\mathrm{Cl}(0, 2D)$ anticommutation at every D, Walsh-to-spherical multiplicity flow, and exact low-mode commutator-gap on Walsh-weight ≤ 1 modes; residual reduced to one uniform Sobolev-tail estimate (61).
10	**Empirical M_* bounds.**
§10.8	SMEFT operator-by-operator survey of $M_* = 4\pi m_h \sqrt{2/3}$ at PST's tree level vs LHC/EWPO bounds; $\Delta S_{\mathrm{PST}} = \Delta T_{\mathrm{PST}} = 0$ structurally.
11	**Bimodule support: threshold-selected $\mathrm{SU}(2)_L$ acts on the $\gamma_F = +1$ block only.**
§8.21	Closes §14.2 (B), the chiral-bimodule residual left open by Computation 8. Constructs a minimal but representative chiral H_F: one weak doublet $((\nu_L, e_L), (\nu_R, e_R))$ with $\gamma_F = \mathrm{diag}(+1, +1, -1, -1)$. Two representations of A_F are mathematically possible *a priori*: Choice A ($\mathbb{H}$ acts on $H_F^{(+)}$ left block, trivial on $H_F^{(-)}$ – the parity-violating observed SM); Choice B ($\mathbb{H}$ acts on $H_F^{(-)}$ right block, trivial on $H_F^{(+)}$ – the parity-conjugate model, never observed). Both give consistent spectral triples in the Connes-Chamseddine sense; the chirality of $\mathrm{SU}(2)_L$ therefore reduces to "which of A or B is realised in nature?". PST's claim is that the directed modal threshold (paper §8) is the physical principle that selects Choice A. Verifies (i) both representations of $\mathbb{H}$ commute with γ_F (necessary condition); (ii) both have clean block support; (iii) the substrate-side $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$ operator-norm signatures under the threshold $\varepsilon > 0$ break the L↔R symmetry as required, selecting Choice A.

Comp.	Result
12 §8.7	**Consistency of octonionic-**SU(3) **embedding with Computations 3–T.** Examines whether the octonionic-SU(3) embedding ($\mathrm{SU}(3) \subset G_2 = \mathrm{Aut}(\mathbb{O})$ acting as automorphisms preserving the octonion product) is consistent with the rest of PST's structure, vs. the canonical block embedding ($\mathrm{SU}(3) \subset \mathrm{SU}(4) \cong \mathrm{Spin}(6)$ acting as block $\mathrm{diag}(\mathrm{SU}(3), 1)$ on the half-spinor). If the octonionic interpretation is consistent with Computations 3–T, Furey's three-generation construction ports directly into PST without changing framework assumptions. Six structural checks: (§1) two SU(3) subgroups of Spin(6) compared; (§2) Computation 6's gauge group derivation does not specify which SU(3); (§3) Computation 7's order-one transfers under either embedding; (§4) inner fluctuations and the Higgs sector are unaffected; (§5) Computation 4's spin structure and Computation 9's convergence are independent of the embedding; (§6) verdict: octonionic interpretation is structurally consistent with PST's existing results.
13 §8.7	**Canonical vs octonionic** SU(3) **side-by-side comparison.** Computes the consequences of canonical vs octonionic SU(3) embedding by re-deriving Computation 1 §7 (inner-fluctuation extension of the Yukawa structural-scope theorem) and Computation 7 (order-one condition selecting Yukawa form) under both, then compares for "best PST fit". Best-fit criteria in priority order: (C1) does $N_{\mathrm{gen}} = 3$ emerge structurally? (C2) do existing PST results survive – chirality, Yukawa contingency, empirical bounds, gauge group? (C3) is the construction algebraically clean – no new posits? (C4) consistency with the Connes-Chamseddine framework? Seven sections culminate in the recommendation of the octonionic embedding as the natural source of $N_{\mathrm{gen}} = 3$ via Furey's six-SU(3)-triplet decomposition of $\mathrm{Cl}(0, 6)$, with the open work being the explicit chain-algebra construction (Computations 14–17).

Comp.	Result
14 §8.7	**Inner-fluctuation structural-scope theorem extension (Computation 1 §7) under octonionic embedding.** Step B of the octonionic-SU(3) re-derivation programme identified by Computation 13. Original Computation 1 §7 (canonical embedding) assumed $H_F = H_F^{(1)} \otimes \mathbb{C}^{N_{\text{gen}}}$ tensor-factor structure for generations, with A_F acting as $a^{(1)} \otimes I_{\text{gen}}$ (generation-blind), so $[D_F, b]$ inherits generation structure of D_F unchanged. Under the octonionic embedding, H_F is NOT in tensor-factor form for generations; the three "generation copies" are internal states of the 64-complex-dim algebra $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$, with A_F's $M_3(\mathbb{C})$ acting via the octonionic SU(3), mixing generation copies by SU(3) rotations. The canonical §7 argument "A_F is gen-blind" does NOT transfer. The substantive question is whether the structural-scope STATEMENT itself still holds: can inner fluctuations turn a gen-symmetric $D_F^{(0)}$ into one with generation-asymmetric Yukawa eigenvalues? Five sections: (§1) toy octonionic-SU(3) model; (§2–§3) inner fluctuations under each embedding; (§4) substrate-level S_3 constraint as the deeper invariant; (§5) verdict: individual fluctuations can mix generations but substrate S_3 invariance constrains physical observables; Yukawa contingency preserved.
15 §8.7	**Order-one Yukawa selection (Computation 7) under octonionic embedding.** Step C of the octonionic re-derivation programme. Verifies Computation 7's result "order-one condition selects the Yukawa form" under the octonionic A_F action and determines the specific Yukawa form selected. Original Computation 7 (canonical): H_F = 8-dim lepton-sector finite triple, A_F acts via canonical lepton-sector representation ($M_3(\mathbb{C})$ trivial on leptons, $\mathbb{H}$ on left doublet), order-one $[[D_F, a], Jb^*J^{-1}] = 0$ SELECTS the Yukawa form of D_F. Under the octonionic embedding, A_F's action on H_F changes (the chain-algebra structure of $\overleftarrow{\mathbb{O}}$), so the order-one condition is a different algebraic constraint. Two possible outcomes: (a) order-one still selects A Yukawa form (possibly different from canonical) – supports the octonionic interpretation; (b) order-one is inconsistent with any Yukawa form – falsifies octonionic interpretation. Seven sections culminate in the verdict: order-one selecting condition is unitarily covariant; Yukawa residue $\sim 2 \times 10^{-5}$ vs ~ 10 for generic D_F under BOTH embeddings – the selecting condition transfers, and the octonionic interpretation passes this test.

Comp.	Result
16 §8.5	**Furey foundation: $\overleftarrow{\mathbb{O}} \cong \mathbb{C} \otimes \mathrm{Cl}(0,6) \cong M_8(\mathbb{C})$, octonionic SU(3) as G_2-stabiliser.** The most ambitious computational physics in this track series: attempts to carry out Furey's 2014 construction explicitly inside PST's substrate Clifford algebra. Eight sections: (§1) octonion multiplication table (Fano-plane convention); (§2) left-multiplication operators L_{e_a} as 8×8 complex matrices; (§3) verify the algebra structure of $\overleftarrow{\mathbb{O}}$ as 64 explicit 8×8 matrices; (§4) identify the primitive idempotent f (vacuum) of $\overleftarrow{\mathbb{O}}$; (§5) construct the octonionic $\mathrm{SU}(3) \subset G_2$ as stabiliser of f and verify the 8 adjoint elements with exact commutation relations; (§6) decompose $\overleftarrow{\mathbb{O}}$ (64-complex-dim) under this SU(3) into irreducible representations; (§7) compare to Furey's $16 + 48$ split; (§8) honest assessment of how rigorously this verifies the three-generation interpretation. Verifies $\overleftarrow{\mathbb{O}} \cong \mathbb{C} \otimes \mathrm{Cl}(0,6) = M_8(\mathbb{C})$ and octonionic SU(3) as G_2-stabiliser of e_7 to machine precision. The matter-sector multiplicity is off by 6 due to degenerate Cartan weights; resolved by Casimir-discrimination in Computation 17.
17 §8.6	**Definitive SU(3) decomposition of $M_8(\mathbb{C})$ via Casimir.** Closes the off-by-6 residue from Computation 16 by using the SU(3) quadratic Casimir $C_2 = \sum_a (\Lambda_a)^2$ as an irrep discriminator. Computation 16 identified the eight-generator octonionic SU(3) subgroup (8 adjoint elements verified, commutation relations exact), but the matter-sector multiplicity count was off because the joint (Λ_3, Λ_8) eigenbasis is not unique on degenerate weights – singlets and $(0,0)$ members of triplets share Cartan weights and can only be told apart by the Casimir. Six sections: (§1) reconstruct the octonionic SU(3) generators (lifted from Gell-Mann via the $J = L_{e_7}$ complex structure on $(e_1, e_3), (e_2, e_6), (e_4, e_5)$ – the correct Cayley pairing for the G_2-stabiliser of e_7); (§2) build $\mathrm{ad}(\Lambda_a)$ on $M_8(\mathbb{C})$; (§3) build the Casimir C_2 as a 64×64 Hermitian matrix; (§4) diagonalise C_2 and group eigenvalues by SU(3) irrep Casimir values (0 for $\mathbf{1}$, 4/3 for $\mathbf{3}$, 3 for $\mathbf{8}$, 10/3 for $\mathbf{6}$); (§5) confirm the decomposition $6 \cdot \mathbf{1} + 5 \cdot \mathbf{3} + 5 \cdot \overline{\mathbf{3}} + 2 \cdot \mathbf{8} + \mathbf{6} + \overline{\mathbf{6}}$ (total dim 64) matches the Clebsch–Gordan expectation to machine precision; (§6) status statement for §14.2 (C): octonionic SU(3) structurally derived, decomposition exact.
18 §8.8	**ω as the unique Spin(6) singlet.** Spin(6)-invariance in $\mathrm{Cl}(0,6) \otimes \mathbb{C}$: $\omega = L_{e_1} \ldots L_{e_6}$ is the unique (up to scalar) Spin(6)-invariant element of positive grade. Hence $f = (I + i\omega)/2$ is the unique (up to chiral swap) Spin(6)-invariant primitive idempotent, exactly Furey's idempotent.

Comp.	Result
19	**Substrate parity grading = $\mathrm{Cl}(0,6)$ chirality grading.**
§8.8	Schur on the irreducible chiral halves $H_\pm$ of $H_F = \mathbb{C}^8$ gives Hermitian Spin(6)-commutant = span$\{I, \gamma\}$ (2-real-dim). In the JW Majorana representation, Computation 3's parity grading $\gamma_K = Z^{\otimes D}$ equals $i\omega_{\mathrm{JW}}$ exactly ($\|\gamma_K - i\omega_{\mathrm{JW}}\| = 0$ to machine precision). The chirality projector $f = (I + i\omega)/2$ is delivered directly by the substrate's parity-selected real structure; this is Furey's primitive idempotent [120], and her acting-on-$\mathbb{C} \otimes \mathbb{O}$ chain-algebra construction then delivers $\mathrm{SU}(3) \times \mathrm{U}(1)$ on one generation under $\mathrm{SU}(3)_c$. The synthesis to the full $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ requires the $\mathbb{H}$ factor from the emergent Lorentzian Dirac sector $\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ (Computation 66) combined via Dixon's hyperspinor framework $T = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ [144].
Dirac-convergence step (§7.8): L_round and L_comm computations.	
20	**L_round sweep.**
§7.8	PST analog of [118] Lemma 3.2 across $D = 4, 6, \ldots, 20$ and three test classes under eleven candidate Lip-norms. All polynomial Lip-norms FAIL; exponential Lip-norms $e^{c\|S\|}$ with $c > \log 2$ succeed at rate $\gamma_D = O((2/e^c)^D)$. Uses fast Walsh–Hadamard transform.
21	**Dirac commutator Lip-norm scaling.**
§7.8	Measures $\|[D, \chi_S]\|_{\mathrm{op}}$ for the discrete Dirac $D = \sum_a \chi_a$ on Walsh modes χ_S; result $= 2\sqrt{\|S\|}$ exactly at every D and every $\|S\|$, matching the Friedrich Dirac eigenvalue scaling on round S^3 [112]. The first-order Dirac structure on the substrate is polynomial (not exponential) in Walsh weight.
22	**Higher-order Dirac commutators.**
§7.8	$\mathrm{ad}_D^k(\chi_S)$, $k = 1, \ldots, 10$. Result: $\|\mathrm{ad}_D^k(\chi_S)\|^{1/k} \to 2\sqrt{D}$ (independent of $\|S\|$ at large k). The spectral radius of ad_D is $2\sqrt{D}$, so the Fréchet smooth Lip-norm $L_F(\chi_S) = \sup_j \|\mathrm{ad}_D^j(\chi_S)\|/j! \sim e^{2\sqrt{D}}$ by Stirling.
23	**Fréchet smooth L_round closure.**
§7.8	Measures L_F from equation (63) on the constant-coefficient Walsh tail. Result: $\gamma_D \approx 0.37 \cdot e^{-0.28D}$ at $D = 4, 6, 8, 10, 12$, exponential decay matching the natural spectral-triple smooth subalgebra. With [115] Theorem 5.2 this gives QGH convergence of the Walsh substrate algebras to $C(S^3)$.

Comp.	Result
24	**χ_a–χ_S discrete-side audit.**
§7.8	Confirms (i) $\|[\chi_a, \chi_S]\|_{\text{op}} = 0$ if $a \notin S$, exactly 2 if $a \in S$; (ii) the Hilbert–Schmidt overlap of $[\chi_a, \chi_S]$ with every Walsh mode χ_T is zero to machine precision. The bridge defect lives entirely in the non-abelian σ_y/σ_x Pauli sector.
25	**Friedrich S^3 Dirac spectrum.**
§7.8	Eigenvalues $\pm(n+3/2)/\ell$, multiplicity $2(n+1)(n+2)$ ([112]). Cross-checks the naive scalar-spinor count $2(l+1)^2$, exhibiting the Pauli mixing factor $(n+2)/(n+1)$ from Friedrich's eigenspinor decomposition ([113]). Continuum-side reference for the spinor extension of b_D.
26	**Bijective-bridge no-go.**
§7.8	Structural no-go ruling out the entire bijective single-mode bridge class for L_comm. The discrete-side commutator norm is exactly $\|[\chi_a^{\text{Cliff}}, \chi_S]\|_{\text{op}} = 2$ for $a \in S$ (zero otherwise), independent of D and of $\|S\|$ (verified at $D \in \{4,6,8\}$, $\|S\| \in \{1,2,3\}$, machine precision). The continuum-side norm scales as $\sqrt{l(l+1)}$ via the SU(2) Casimir. The L_round scaling on the same bridge (Computations 20, 25) fixes the weight-graded scalar, leaving no freedom to reconcile the two scales. Conclusion: no bijective single-mode bridge closes L_comm.
27	**Bergman/Toeplitz framework on $H^2_\alpha(B^2)$.**
§7.8	Full operator-theoretic implementation of [118]. The Toeplitz commutator $[T_{\bar{z}_1^k}, T_{z_1^k}]$ is diagonal on the monomial basis with explicit entry $\Lambda(J,k,\alpha) = (m_1+1)_k/(\|J\|+\alpha+3)_k - (m_1)_{(k)}/(\|J\|+\alpha+3-k)_k$. At $k=1$, $\|[T_{\bar{z}}, T_z]\|_{\text{op},\alpha} = 1/(\alpha+3)$ *exactly* (machine-precision match against the analytical formula, J-maximizer $(0,0)$); at $k=2$ the commutator norm is 0.30 at $\alpha=0$, 0.20 at $\alpha=1$, 0.085 at $\alpha=5$. The Toeplitz operator norm $\|T_{z_1^k}\|_{\text{op},\alpha}$ approaches 1 and is α-INDEPENDENT at leading order; the ratio $R(k,\alpha)$ is α-DEPENDENT ($R(1,-0.5) = 0.40$ vs $R(1,10) = 0.08$). With $\alpha = \alpha(D)$ fidelity and gap rate can be tuned simultaneously.

Comp.	Result
28	**Explicit substrate–Toeplitz bridge probe (D = 4).**
§7.8	First explicit instantiation of the candidate substrate ↔ Toeplitz bridge at $D = 4$, $N = 5$. Sites $0, 1$ of the substrate map to T_{z_0}, T_{z_1}; sites $2, 3$ map to $T_{\bar{z}_0}, T_{\bar{z}_1}$. Single-mode bridge images have op-norm ~ 0.84 at $\alpha = 0$, decreasing with α as predicted. Commutator-gap probe across six (a, S) pairs from Computation 26: cases where $a \in S$ at single sites map to zero on the Toeplitz side (multiplicative bridge with holomorphic generators commutes); mixed holomorphic/anti-holomorphic cases produce non-zero bridge commutators (0.193 at $\alpha = 0$ for $S = \{0, 2\}$). Conclusion: the multiplicative bridge is a necessary scaffolding but not sufficient. It correctly maps algebraic generators but does not preserve the abelian Walsh structure: spurious cross-commutators appear at weight $k \geq 2$, scaling as $O(1/\alpha)$ at large α (consistent with Computation 27's prediction). The next concrete step for closing the Dirac-aware lift is the Mosco-limit reweighting at each weight k: map a SUM of Walsh modes to a single degree-l Toeplitz symbol $Y^l(z, \bar{z})$, with coefficients chosen to suppress the spurious cross-commutators while preserving the α-dependent gap-rate scaling.
29	**Mosco-limit reweighting via holomorphic Toeplitz subalgebra.**
§7.8	Restricts the bridge image of Computation 28 to the holomorphic Toeplitz subalgebra of $H^2_\alpha(B^2)$, where all operators commute (multiplication by holomorphic functions on H^2_α is just multiplication, no projection needed). Site-to-holomorphic mapping at $D = 4$: sites $0, 1 \to T_{z_0}, T_{z_1}$; sites $2, 3 \to T_{z_0^2}, T_{z_1^2}$. Bridge of χ_S = product of individual-site images, in the abelian subalgebra. Sanity check: all bridge-image cross-commutators $[b(\chi_a), b(\chi_b)]$ are zero to machine precision (the abelian Walsh structure is preserved by construction, eliminating the spurious cross-commutators of Computation 28). Bridge image norms scale with weight: 0.85 at $k = 1$, 0.38 at $k = 2$, 0.17 at $k = 3$ (truncation collapse at $k = 4$ at $N = 5$). Gap probe: substrate commutator norm 2 (when $a \in S$) maps to Toeplitz commutator ~ 0.71 at $\alpha = 0$, decaying with α; cases with $a \notin S$ (substrate norm 0) still give residual non-zero Toeplitz commutators (~ 0.43 at $\alpha = 0$). Conclusion: holomorphic Toeplitz subalgebra is the structurally right target (abelian preservation), but the simplest Clifford analog $T_{\bar{z}_a}$ still produces residuals in the "$a \notin S$" sector. The next concrete step is constructing the α-twisted Berezin derivative $D_\alpha = \sum_a (T_{\bar{z}_a} + \alpha\text{-correction})$ designed to make $[D_\alpha, b(\chi_S)]$ vanish for $a \notin S$ while preserving the α-dependent gap rate.

Comp.	Result
30	**Optimal-α closure diagnostic for the holomorphic bridge.**
§7.8	Tests whether a single $\alpha = \alpha(D)$ can match the substrate commutator across all (a, S) test cases simultaneously, given the abelian-preserving holomorphic bridge of Computation 29. At $D = 4$, $N = 5$, sweep $\alpha \in [-0.5, 30]$ across eight test cases. *Finding*: per-case optimal α values fall into two clusters: cases with substrate norm = 2 ($a \in S$) want $\alpha = -0.5$ (boundary of the BS family), cases with substrate norm = 0 ($a \notin S$) want $\alpha = 30$ (deep into the ball). Spread of optimal α values is 30.5 – no single α closes both sectors. The aggregate optimal $\alpha = -0.5$ gives residual ~ 8.9. *Structural observation*: the substrate commutator is BIMODAL (0 or 2), but the Toeplitz commutator under the simple $T_{\bar{z}_a}$ Clifford analog is a CONTINUOUS function of α; no scalar α can produce a bimodal output. Closing the Dirac-aware lift therefore requires additional internal degrees of freedom in the Clifford construction beyond a single α: an α-twisted Berezin derivative with selectivity per substrate site, or the explicit spinor extension on a tensor $\mathbb{C}^2$ factor.
31	**Spinor extension on $H^2_\alpha(B^2) \otimes \mathbb{C}^N$ for L_comm.**
§7.8	Tests whether tensoring a spinor $\mathbb{C}^2$ or $\mathbb{C}^4$ fibre onto the Bergman space supplies the internal degrees of freedom identified as missing by Computation 30. Three constructions tested at $D = 4$, $N = 5$: (A) trivial $\mathbb{C}^2$ spinor with σ-factor on Clifford bridge, (B) $\mathbb{C}^4$ spinor with full $\mathrm{Cl}(0, 4)$ generators tensored with HOLO Toeplitz, (C) mixed bridge with ANTI-HOLO Toeplitz tensored with $\mathrm{Cl}(0, 4)$ generators. *Findings*: in (A) and (C) the tensor factor decouples in the operator norm (op-norm of $A \otimes B$ factorises as $\|A\| \cdot \|B\|$); the spinor factor multiplies the existing gap by a unitary, leaving the gap structure unchanged from Computations 29/36. In (B) all cross-commutators are exactly ZERO because both bridge images sit in the abelian holomorphic Toeplitz subalgebra. Conclusion: tensor spinor extensions do NOT close L_comm on their own. Closing it requires a NON-TENSOR coupling between the Bergman and spinor factors. The natural candidate is the round-S^3 Dirac $D = \sum_a \sigma_a \otimes X_a$, where X_a are the SU(2) left-invariant vector fields on H^2_α acting as raising/lowering operators on the Wigner-D monomial basis.

Comp.	Result
32	**SU(2) Dirac operator on $H^2_\alpha(B^2) \otimes \mathbb{C}^2$ for L_comm.**
§7.8	Implements the non-tensor coupling identified as missing by Computation 31: the round-S^3 Dirac $D = \sum_a \sigma_a \otimes J_a$, where J_a are SU(2) generators acting on the Bergman monomial basis as raising/lowering operators ($J_+ = z_1\partial_{z_2}$, $J_- = z_2\partial_{z_1}$, $J_z = \frac{1}{2}(z_1\partial_{z_1} - z_2\partial_{z_2})$). Verifies su(2) algebra to machine precision and Hermiticity of D ($\|D - D^*\| < 10^{-14}$); eigenvalue range $[-3.5, 2.5]$ at $N = 5$ approximates Friedrich's $\pm(n + 3/2)$ S^3 spectrum. *Key finding*: the L_comm gap $\|[D, b_{\text{holo}}(\chi_S) \otimes I]\|_{\text{op}}$ is NON-ZERO – the non-tensor coupling finally produces a meaningful Dirac commutator. Normalised by bridge fidelity, the ratios are 1.02 at $\|S\| = 1$, 1.83–2.29 at $\|S\| = 2$, 2.45–3.04 at $\|S\| = 3$, approaching the substrate target $2\sqrt{\|S\|}$ but with variance across same-weight S. The ratios are α-independent (the J_a structure on the ONB is α-canceling), meaning α-dependence enters only through the bridge image norms. Conclusion: the SU(2) Dirac framework is operational and produces the right qualitative L_comm structure; quantitative closure requires (i) Mosco-limit averaging across same-weight Walsh modes to remove the variance, and (ii) α-dependent rescaling to match $2\sqrt{\|S\|}$ exactly.
33	**Mosco averaging and α rescaling for L_comm closure.**
§7.8	Combines the two within-framework tunings identified by Computation 32. *Mosco averaging*: $b_{\text{Mosco,k}} := (1/\sqrt{C(D,k)}) \sum_{\|S\|=k} b_{\text{holo}}(\chi_S)$, collapsing $C(D,k)$ weight-k Walsh modes into one symmetric combination. *α rescaling*: $\text{bridge}_{\text{final,k}} := b_{\text{Mosco,k}}/\|b_{\text{Mosco,k}}\|_{\text{op}}$ (unit op-norm per weight class). Tested at $D = 4$, $N = 5$. *Findings*: Step 3 ratios $\|[D_\alpha, \text{bridge}_{\text{final,k}} \otimes I]\|_{\text{op}}/(2\sqrt{k})$ monotonically approach 1 with weight: 0.60 ($k = 1$), 0.71 ($k = 2$), 0.87 ($k = 3$). The L_comm gap is closing asymptotically. $k = 4$ is zero due to truncation collapse ($N = 5$ cannot hold the full-weight product). *Residual*: Step 4 variance check shows per-S ratios at fixed $\|S\|$ still range 1.02–1.45 at $k = 1$ even after rescaling, indicating the symmetric-sum Mosco is approximate. Exact uniformity would require a Wigner-D-component projection within each weight class. Conclusion: this is the FIRST computation demonstrating monotone convergence of the L_comm ratio toward closure in the operational framework. Closing the residual variance requires the proper Wigner-D-decomposed Mosco averaging; closing the truncation effect requires $N \geq 2D$. Both are within-framework refinements, not structural changes.

Comp.	Result
34 §7.8	**L_comm convergence at larger (D, N): asymptotic closure test.** Scales Computation 33 up to $D = 4$, $N = 10$ and $D = 6$, $N = 12$ to test whether the Mosco-averaged α-rescaled bridge ratios converge to 1. *Findings* at $\alpha = 0$: ratios increase with (D, N) at fixed k (e.g., $k = 3$: $0.86 \to 0.81 \to 0.96$ across $(4, 5)$, $(4, 10)$, $(6, 12)$), and increase with k at fixed (D, N). The $k = 3$ ratio at $D = 6, N = 12$ reaches 0.96 – within 4% of substrate target $2\sqrt{3}$. *Crossover observation*: at $D = 6, N = 12$, the ratio crosses 1 around $k = 3$–4 and continues growing: 1.12 ($k = 4$), 1.23 ($k = 5$), 1.32 ($k = 6$). This is structurally explained: the SU(2) Dirac has 3 frame directions (giving $\sim \sqrt{3} \cdot \|J_a\|$ scaling), while the substrate has D Clifford directions (giving $2\sqrt{\lvert S\rvert}$). These match at $k \sim 3$ but diverge at higher $\lvert S\rvert$. *Important framing*: this test compares $\|\text{bridge commutator}\|$ to $\|\text{substrate commutator}\|$, but the genuine L_comm criterion is the HOMOMORPHISM GAP $\|\text{bridge}([D, \chi_S]) - [D_\alpha, \text{bridge}(\chi_S)]\|_{\text{op}} \to 0$, which is a stronger condition. The norm-match closure at $k \sim 3$ is suggestive but the homomorphism-gap test is the next concrete deliverable.
35 §7.8	**L_comm homomorphism-gap test (explicit construction).** Tests the genuine L_comm criterion: $\|\text{bridge}([D_{\text{sub}}, \chi_S]) - [D_\alpha, \text{bridge}(\chi_S)]\|_{\text{op}}$. Uses the natural multiplicative bridge extension $\text{bridge}(\chi_a^{\text{Cliff}}) := J_a \otimes \sigma_a$ (with cyclic Pauli assignment for the substrate-vs-Pauli site mismatch). Computes $\text{LHS} = 2\sum_{a \in S}(J_a\, b_{\text{holo}}) \otimes \sigma_a$ and $\text{RHS} = [D_\alpha, b_{\text{holo}} \otimes I]$ directly, then their operator-norm difference. *Negative finding*: the gap/LHS ratio is large (0.68 to 1.01 at $D = 4, N = 5$; 0.83 to 1.00 at $D = 4, N = 10$; 0.82 to 1.00 at $D = 6, N = 12$) and does NOT shrink with increasing (D, N). The bridge with the natural multiplicative extension is therefore NOT a Dirac homomorphism, and the failure is uniform across truncations. *Structural reading*: the earlier norm-match successes (Computation 34 ratio 0.96 at $k = 3$) measured a WEAKER quantity than L_comm. Closing the true homomorphism gap requires either a different bridge construction (e.g., spinor fibre $\mathbb{C}^{2^D}$ matching the substrate Cl(0, 2D) representation rather than the round-S^3 $\mathbb{C}^2$) or a re-interpretation of L_comm in terms of norm-equivalence-up-to-rescaling rather than exact operator equality.

Comp.	Result
36 §7.8	**L_comm finite-N matched-truncation convergence at $D = 8$, $N = 20$.** Extends Computation 34 to substrate size $D = 8$ with Bergman truncation $N = 20$ (Bergman dim 231, full Hilbert dim 462). Measures the relaxed L_comm gap $\gamma_D(k) = \lvert 1 - \text{ratio}\rvert$ where ratio $= \|[D_\alpha, \text{bridge}_{\text{final},k} \otimes I]\|_{\text{op}}/(2\sqrt{k})$. *Finding (finite-$N = 20$)*: at the Walsh-weight cutoff $k = k_D = \lfloor\sqrt{D}\rfloor = 2$, the ratio reaches 0.993 ($\gamma_D = 0.007$) at $\alpha = 0$ – within 0.7% of the substrate target. Compared to $\gamma_D = 0.285$ at $D = 4$, $\gamma_D = 0.203$ at $D = 6$: linear fit $\log(\gamma_D) = -0.93D + 2.96$, i.e. $\gamma_D \sim 19.3 \cdot e^{-0.93D}$, empirical *finite-$N$* exponential decay with rate $c \approx 0.93$. *Caveat established by Computations 37–38*: the rate 0.93 is specific to the matched truncation $N(D) \sim 2.5\,D$; the infinite-N analytical rate at $k_D = 2$ is $c \approx 0.527$. Ratios above the cutoff ($k > k_D$) diverge as expected. With the paper-side commitment to the relaxed criterion and the finite-N verification both done, the infinite-N closure is supplied by the refined symmetric-monomial bridge of Computations 39–41.
37 §7.8	**Analytical-numerical bridge validation at $D = 4$, $k = 1$.** Sets up the closed-form infinite-N framework: the Leibniz identity $[J_a, T_F] = T_{J_a F}$ for the SU(2) generators acting on holomorphic Toeplitz symbols on $H^2_\alpha(B^2)$. The op-norm $\|[D_\alpha, T_F \otimes I_{\mathbb{C}^2}]\|_{\text{op}}$ at infinite Bergman truncation reduces to $\sup_{\partial B^2} \|M(z)\|_{\text{op},2\times 2}$ where $M(z) = \sigma_a J_a F(z)$. The correct pointwise op-norm for complex a, b, c is $\|\sigma_1 a + \sigma_2 b + \sigma_3 c\|^2_{\text{op}} = (\lvert a\rvert^2 + \lvert b\rvert^2 + \lvert c\rvert^2) + \sqrt{X^2 + \lvert Y\rvert^2}$ with $X = (\lvert J_+ F\rvert^2 - \lvert J_- F\rvert^2)/2$ and $Y = 2i\,\text{Im}(\overline{J_z F} \cdot J_- F)$; the naive formula $\sqrt{\lvert a\rvert^2 + \lvert b\rvert^2 + \lvert c\rvert^2}$ misses the $\sqrt{X^2 + \lvert Y\rvert^2}$ term for complex symbols. *Validation*: at $D = 4$, $k = 1$, $\alpha = 0$, $N = 40$, the matrix singular values give $\|T_F\|_{\text{op}} = 1.156$ (analytical 1.207) and $\|[D_\alpha, T_F \otimes I]\|_{\text{op}} = 1.466$ (analytical 1.532). The ratio matches the analytical sup-formula to four digits: 0.6344 numerical vs 0.6347 analytical. The framework is therefore an explicit constrained-optimization sequence over ∂B^2 indexed by (D, k).

Comp.	Result
38	**Infinite-N L_comm gap at $k = k_D$: cutoff-transition discontinuity.**
§7.8	Applies the validated framework (Computation 37) to compute the infinite-N analytical gap $\gamma_D^{(\infty)}$ at the Walsh-weight cutoff $k = k_D$ for $D = 4, 6, 8, 10, 12$, by grid-searching sup over the 3-real-parameter unit sphere in $\mathbb{C}^2$. *Findings*: (i) at fixed $k_D = 2$ ($D = 4, 6, 8$), the analytical gap is $\gamma_D^{(\infty)} = 0.389, 0.193, 0.048$, fitting an exponential at rate $c \approx 0.527$ – not the finite-N rate 0.93 of Computation 36. The difference is supplied by the matched-truncation correction $\gamma_D^{(\infty)} - \gamma_D^{(N=20)}$. (ii) At the cutoff transition $k_D = 2 \to 3$ (between $D = 9$ and $D = 10$), the analytical gap JUMPS from 0.048 to 0.444 – the closure-at-k_D structure is not uniform in D at infinite N under the multiplicative bridge of Computations 32–36. The discontinuity is eliminated by the refined symmetric-monomial bridge of Computations 39–41.
39	**Refined symmetric-monomial bridge: closed-form ratio.**
§7.8	Constructs the symmetric bridge $b_C(\chi_S) := T_{(z_0 z_1)^{m(\lvert S\rvert)}}$ replacing the multiplicative Mosco-averaged bridge of Computations 32–36. All Walsh modes of the same weight $\lvert S\rvert = k$ get mapped to the same operator, so Mosco averaging is trivial. Closed-form ratio: $\sup \lVert M\rVert / \sup \lvert F\rvert = (m+1)\sqrt{(1-1/m^2)^{m-1}}$, with the sup attained at $r^2 - s^2 = 1/m$ on ∂B^2. At $m = 1$: ratio $= 2 = 2\sqrt{1}$, *exact* closure at $k = 1$ with zero parameters. For integer $m(k)$ chosen to minimize gap, the residual is 0.02–0.11 across $k = 1$–11.
40	**Two-monomial refined bridge: even-k closure.**
§7.8	Adds a tunable parameter via the superposition $f_k = (z_0 z_1)^{m(k)} + \alpha(k)(z_0 z_1)^{m(k)+1}$. Numerical grid search yields essentially exact closure (gap $< 10^{-3}$) at $k = 1, 2, 4, 6, 8$ (i.e., even k and $k = 1$): $\alpha(2) = -0.745$ at $m = 2$; $\alpha(4) = -1.777$ at $m = 3$; $\alpha(8) = -0.308$ at $m = 5$. Odd $k = 3, 5, 7$ retain residual gap 0.05–0.09 in the two-monomial family.
41	**Three-monomial refined bridge: odd-k closure.**
§7.8	Extends to $f_k = (z_0 z_1)^{m(k)-1} + \beta(k)(z_0 z_1)^{m(k)} + \alpha(k)(z_0 z_1)^{m(k)+1}$ with two free parameters per k. Grid search closes odd k: at $k = 3$, $m = 3$, $\beta = -3.29$, $\alpha = 3.19$, ratio $= 3.46410 = 2\sqrt{3}$ *exactly*; at $k = 7$, $m = 5$, $\beta = -1.51$, $\alpha = 2.00$, ratio $= 5.29151$, gap $= 1.5 \times 10^{-5}$. Combined with Computation 40, the three-monomial refined bridge closes L_comm with $\gamma(k)$ *D-independent* and below 10^{-4} at all confirmed k, eliminating the k_D cutoff-transition discontinuity exposed in Computation 38. The infinite-N closure of the relaxed criterion is therefore demonstrated achievable; the rigorous proof is the remaining analytical step.

Comp.	Result
42	**Refined-bridge closure verification at higher k.**
§7.8	Computation 41 had a verification gap at $k = 5$ (search converged to a local minimum) and did not test $k > 7$. This script uses a multi-restart grid search with eleven seed points to escape local minima. Results: $k = 5$: $m = 4$, $\beta = -2.92$, $\alpha = -3.50$, ratio $= 4.47214 = 2\sqrt{5}$ *exactly*; $k = 9$: $m = 6$, $\beta = 1.53$, $\alpha = -1.50$, gap $= 3.6 \times 10^{-4}$; $k = 10$: $m = 6$, $\beta = 3.91$, $\alpha = -2.50$, gap $= 5 \times 10^{-5}$; $k = 11$: $m = 6$, $\beta = -0.44$, $\alpha = 5.24$, gap $= 8 \times 10^{-5}$; $k = 12$: $m = 6$, $\beta = -1.29$, $\alpha = -9.52$, ratio $= 6.92820 = 2\sqrt{12}$ *exactly*. Combined with Computations 39–41, the refined symmetric-monomial bridge closes L_comm at every k from 1 to 12 verified, with maximum gap 3.6×10^{-4} at $k = 9$ and exact closure at $k = 1, 3, 5, 12$. The closure pattern continues uniformly with no cutoff-transition discontinuities.
43	**Dixon-algebra foundation: octonion chain algebra on $\mathbb{R}^8$.**
§8.19	Sets up the Fano-plane octonion multiplication explicitly and verifies non-associativity $(e_1e_2)e_4 \neq e_1(e_2e_4)$. Left- and right-multiplication algebras $L_{\mathbb{O}}, R_{\mathbb{O}}$ each fill $M_8(\mathbb{R})$ (matrix rank 64). Centre of $R_{\mathbb{O}}$ is 1-dim (trivial) – no non-trivial central idempotents. Foundational octonion structure used in Computations 53, 57–59.
44	**$\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ multiplication setup.**
§8.19	Builds the multiplication table for the 64-dim real algebra $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, with 11 imaginary units (i_C, i_H, j_H, k_H, $e_1, \ldots, e_7$) each squaring to -1. Non-associativity inherited from $\mathbb{O}$. Cites the structure result $L_{\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}} \cong M_{16}(\mathbb{C})$ (real dim 512) as the natural ambient chain algebra for the generation decomposition (Computation 48).
45	**Extended chiral idempotent f on $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$.**
§8.19	Constructs the extended chiral primitive idempotent $f = (I + i_C \otimes I_H \otimes \omega_6)/2$ where $\omega_6 = L_{e_1} \cdots L_{e_6}$ is the Cl(0,6) volume element ($\omega_6^2 = -I$). Verifies $f^2 = f$ exactly, rank$(f) = 32$ (half of $\mathbb{R}^{64}$). Five of eleven imaginary-unit L-generators commute with f: i_C, i_H, j_H, k_H, e_7 – matching Furey 2014's $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ structure for one generation. Used in §8.19 to fix the chiral half of each per-generation Hilbert space $\mathcal{H}_k$ at dimension 16.

Comp.	Result
46 §8.19	**Three quaternionic sub-algebras containing fixed $C \subset \mathbb{O}$ (structural seed for $N_{\text{gen}} = 3$).** Fixes $C = \text{span}\{1, e_1\} \subset \mathbb{O}$ and enumerates all quaternionic sub-algebras containing C. Result: exactly 3 – $Q_1 = \{1, e_1, e_2, e_3\}$, $Q_2 = \{1, e_1, e_4, e_5\}$, $Q_3 = \{1, e_1, e_6, e_7\}$ – one per Fano-plane line through e_1. They partition $V_7 \setminus \text{span}\{e_1\} = \text{span}\{e_2, \ldots, e_7\}$ into three disjoint 2-dim pairs. Count of 3 is G_2-symmetric (same for any unit vector in V_7). Combined with the V_7-valued asymmetric tension of P2, this is the combinatorial structure on $\mathbb{O}$ that PST reads as $N_{\text{gen}} = 3$, consistent with Furey 2014's six-SU(3)-triplet decomposition of Cl(0, 6); color-vs-generation distinguishability on $\mathbb{O}$ alone is non-trivial ($\text{Stab}_{G_2}(\hat{\tau}) \cong \text{SU}(3)$ acts on $\hat{\tau}^\perp$ as the fundamental, mixing the three sub-algebras) and is resolved in the ambient $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ (Computation 48).
47 §8.18	**Per-Q_k Cl(0,2) chiral structure.** For each quaternionic sub-algebra $Q_k = \{1, e_1, e_{a_k}, e_{b_k}\}$, verifies $L_{e_{a_k}}, L_{e_{b_k}}$ generate a Cl(0,2) Clifford structure on $\mathbb{R}^8$ with volume $\omega_k^2 = -I$. Each Q_k chiral half is rank 4 on $\mathbb{C}^8$ (4+4 split, structurally identical across all 3). Chiral idempotents $f_k = (I - i\omega_k)/2$ each rank 4, but pairwise products $\|f_i f_j\| = 0.5$ (not mutually orthogonal – they overlap on the same 8-dim space, sharing the generation-blind core).
48 §8.18	**Three disjoint generation sectors in $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$.** Resolves the color-vs-generation distinguishability of $N_{\text{gen}} = 3$ in the Dixon-algebra ambient: the chain-algebra space decomposes as $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} = \text{core} \oplus G_1 \oplus G_2 \oplus G_3$ where $\text{core} = \mathbb{C} \otimes \mathbb{H} \otimes \{1, e_1\}$ (dim 16, generation-blind) and $G_k = \mathbb{C} \otimes \mathbb{H} \otimes \{e_{a_k}, e_{b_k}\}$ (dim 16, generation k). Verifies: $G_i \cap G_j = \emptyset$ for $i \neq j$; $\text{core} + G_1 + G_2 + G_3$ fills the full 64-dim space exactly. Per-generation Hilbert space $\mathcal{H}_k = \text{core} + G_k$ has dim 32; chiral half = 16 states = one full generation of Standard-Model matter (15 SM + 1 right-handed neutrino). Inclusion-exclusion: $3 \times 32 - 3 \times 16 + 16 = 64$ exact.
49 §8.19	**G_2-equivariance verification: $J^2 = -I$ on $\hat{\tau}^\perp$.** For the unit direction $\hat{\tau} \in V_7$ selected at modal sublimation, verifies that the cross-product-induced operator $J\colon \hat{\tau}^\perp \to \hat{\tau}^\perp$ defined by $J(v) := \hat{\tau} \times v$ satisfies $J^2 = -I$ on the 6-dim orthogonal complement $\hat{\tau}^\perp \subset V_7$. This realises the complex structure on $\hat{\tau}^\perp \cong \mathbb{C}^3$ that G_2 preserves on the $\hat{\tau}$-stabilizer $\text{SU}(3)_{\text{col}} \subset G_2$. Underwrites the colour-SU(3)-as-stabilizer mechanism of §8.19 and the disjoint-generation-sector decomposition of Computation 48.

Comp.	Result
50	**Rigorous single-mode L_round closure.**
§7.8	Verifies the elementary bound $\|\mathrm{ad}_D^j(\chi_S)\|_{\mathrm{op}} \le (2\sqrt{D})^j$ across $D \in \{4,5,6,7\}$, $\|S\| \in \{1,2,3\}$, $j \in \{1,\dots,10\}$. Cross-checks $D_{\mathrm{sub}}^2 = D \cdot I$ exactly. The ratio $\|\mathrm{ad}^j\|^{1/j}/(2\sqrt{D})$ approaches 1 from below, reaching 0.93 at $j = 10$. Combined with Stirling, this proves the single-mode L_round closure $\|\chi_S\|_{\mathrm{op}}/L_F(\chi_S) \le e^{-2\sqrt{D}}$ as a rigorous theorem (not merely a numerical observation).
51	**Block-decomposition identity and tail-sum reduction.**
§7.8	Diagonalises D_{sub} via spectral projectors $P_\pm = (I \pm D_{\mathrm{sub}}/\sqrt{D})/2$. Verifies to machine precision that $\mathrm{ad}_{D_{\mathrm{sub}}}$ acts as 0 on diagonal blocks T_{++}, T_{--} and as $\pm 2\sqrt{D}$ on off-diagonal blocks $T_{\pm\mp}$. Establishes the closed-form identity $\|\mathrm{ad}_D^j(T)\|_{\mathrm{op}} = (2\sqrt{D})^j \cdot \max(\|T_{+-}\|, \|T_{-+}\|)$ for every T and every $j \ge 1$. Consequence: $L_F(T) = \max(\|T_{+-}\|, \|T_{-+}\|) \cdot M(D)$ where $M(D) = \sup_j (2\sqrt{D})^j/j!$. For $T_{\mathrm{tail}} = \sum_{\|S\|>k_D} \chi_S$ at $k_D = 2$, $D \in \{4,5,6,7,8\}$: $\|T_{\mathrm{tail}}\|_{\mathrm{op}} = 2^D - \sum_{j \le k_D} \binom{D}{j}$ and $\max(\|(T_{\mathrm{tail}})_{+-}\|, \|(T_{\mathrm{tail}})_{-+}\|) = 2^{D-1} - D$ exactly. True asymptotic L_round rate: $\gamma_D = O(D^{1/4} e^{-2\sqrt{D}})$ (the empirical $0.37 \cdot e^{-0.28D}$ of Computation 23 is a pre-asymptotic fit, crossover near $D \approx 51$). Reduces the remaining analytical work to a closed-form bound on the off-diagonal block $P_+ T_{\mathrm{tail}} P_-$.
52	**Closed-form $\sigma_1 = 2^{D-1} - D$ for the Walsh-tail off-diagonal block.**
§7.8	SVD profile of $P_+ T_{\mathrm{tail}} P_-$ across $D \in \{4,\dots,10\}$ at cutoff $k_D = 2$. The dominant singular value matches $\sigma_1 = 2^{D-1} - D$ exactly at every D; the dominant right singular vector $v_1 = \frac{1}{\sqrt{2}}\big[\|0\rangle - \frac{1}{\sqrt{D}} \sum_a \|e_a\rangle\big]$ is the $-\sqrt{D}$-eigenvector of D_{sub} confined to weights $\{0,1\}$, and the corresponding left singular vector u_1 is its $+\sqrt{D}$ counterpart. Direct calculation in this two-state model yields $\sigma_1 = (\alpha(D) - \beta(D))/2$ with $\alpha(D) = 2^D - 1 - D - \binom{D}{2}$ (the T_{tail} eigenvalue on $\|0\rangle$) and $\beta(D) = -\binom{D-1}{2}$ (on each $\|e_a\rangle$). Simplification gives $\sigma_1 = 2^{D-1} - D$ *exactly*, closing Lemma 2-prime on the S_D-trivial component of $P_+ T_{\mathrm{tail}} P_-$. The second singular value $\sigma_2 = D - 2$ (numerical here; analytically closed in Computations 54, 55) is bounded by an exponential gap from σ_1.

Comp.	Result
53	**Algebraic identity $P_+T_{\text{tail}}P_- = (1/(2\sqrt{D}))\,P_+\,[D_{\text{sub}}, T_{\text{tail}}]$.**
§7.8	Verifies to machine precision the algebraic identity $P_+T_{\text{tail}}P_- = (1/(2\sqrt{D}))\,P_+\,[D_{\text{sub}}, T_{\text{tail}}]$, obtained by substituting $D_{\text{sub}}T_{\text{tail}}D_{\text{sub}} = D\cdot T_{\text{tail}} - D_{\text{sub}}[D_{\text{sub}}, T_{\text{tail}}]$ (which follows from $D_{\text{sub}}^2 = D\cdot I$) into the expansion of $P_+T_{\text{tail}}P_-$. Closed form on the weight-1 standard-rep input alone: $\sigma_{\max} = (D-2)/\sqrt{2} = |\beta(D)-\gamma(D)|/(2\sqrt{2})$ with $\gamma(D) = c(2) = (D-2)(5-D)/2$ the T_{tail} eigenvalue on weight-2 states, achieved via the exact cancellation $D_{\text{sub}}w_2 = D\cdot v$ for $w_2 = \sum_{a<b}(c_b - c_a)|e_a + e_b\rangle$ when $\sum c_a = 0$. (An earlier reading of this script that used the bit-permutation S_D representation reported $\sigma_{\max}$ on a single "axis-1" subspace of the standard rep as 1.63 at $D = 4$, etc.; this is a single direction within the $[D-1, 1]$ block under the wrong rep and does not bound σ_2. The representation that commutes with D_{sub} is the fermionic-signed one; under it $\sigma_2 = D-2$ exactly, Computations 54, 55.)
54	**Identification of the S_D-irrep that saturates $\sigma_2 = D-2$.**
§7.8	Implements the character-projector decomposition of T_{tail}^{+-} under the fermionic-signed S_D representation $\rho_F(g)|S\rangle = \text{sgn}(g \mid S)|g(S)\rangle$ (the naive bit-permutation rep does NOT commute with D_{sub} because of the Jordan-Wigner σ_z strings; verified $\|[\rho_{\text{bit}}(g), D_{\text{sub}}]\|_{\text{op}} \approx 4.0$ at $D = 4$ versus $\|[\rho_F(g), D_{\text{sub}}]\|_{\text{op}} = 0$ to machine precision). Builds $P_\lambda = (\dim\lambda/|S_D|)\sum_g \chi_\lambda(g)\,\rho_F(g)$ for each two-row irrep $[D-j, j]$ with $j = 0, 1, \ldots, \lfloor D/2\rfloor$, then computes $\sigma_{\max}$ within each isotypic block. *Findings:* the standard rep $[D-1, 1]$ saturates $\sigma_{\max} = D-2$ exactly at $D = 4, 5, 6, 7$; every higher two-row irrep gives $\sigma_{\max} = 0$ identically. The multiplicity of $[D-1, 1]$ under ρ_F is 2 (Pieri on $\Lambda^w(\mathbb{C}^D)$ assigns it to H_1 and H_2), so the multiplicity-space matrix M is 2×2; numerically $\text{rank}(M) = 1$ with singular values $(D-2, 0)$.

Comp.	Result
55 §7.8	**Closed-form $\sigma_2 = D-2$ on $[D-1,1]$, completing Lemma 2-prime.** Exhibits the explicit S_{D-1}-fixed vectors $u_1 = \|e_0\rangle - (1/D)\sum_a \|e_a\rangle \in [D-1,1]\cap H_1$ and $u_2 = -\sum_{b>0}\|\{0,b\}\rangle \in [D-1,1]\cap H_2$ and verifies to machine precision (for $D = 4,\dots,8$) the algebraic identities $D_{\text{sub}}u_1 = u_2$, $D_{\text{sub}}u_2 = Du_1$ (no weight-3 leakage; the JW orderings on each triple $\{0,b,c\}$ cancel), $T_{\text{tail}}u_1 = \beta\, u_1$ with $\beta = -\binom{D-1}{2}$, $T_{\text{tail}}u_2 = \gamma\, u_2$ with $\gamma = (D-2)(5-D)/2$. Applying $T^{+-}_{\text{tail}} = (1/(2\sqrt{D}))\, P_+\,[D_{\text{sub}}, T_{\text{tail}}]$ collapses the multiplicity-space matrix in the orthonormal basis $e_1 = u_1/\|u_1\|$, $e_2 = u_2/\|u_2\|$ (with $\|u_1\|^2 = (D-1)/D$, $\|u_2\|^2 = D-1$) to $M = \frac{D-2}{2}\binom{1}{1}\begin{pmatrix}-1 & 1\end{pmatrix} = \frac{D-2}{2}\begin{pmatrix}-1 & +1\\ -1 & +1\end{pmatrix}$, rank 1 with singular values $(D-2,0)$. The saturating right singular vector is $v_\star = (1/\sqrt{2})(-e_1+e_2)$, the left singular vector is $u_\star = (1/\sqrt{2})(e_1+e_2)$. Combined with Computation 52 this gives Lemma 2-prime as a theorem: $\|T^{+-}_{\text{tail}}\|_{\text{op}} = 2^{D-1} - D$ *exactly* for every $D \geq 4$ at $k_D = 2$.
56 §7.8	**Closed-form L_comm ratio for the single-monomial bridge (Lemma 5(a)).** Derives in closed form the L_comm ratio for the single-monomial bridge symbol $f(w) = w^m$ on the holomorphic Toeplitz algebra of the weighted Bergman space ($w = z_0 z_1$). Since $J_z f = 0$ (f is symmetric in z_0, z_1) and $J_\pm f$ are antiholomorphic mirror pairs, M_f is anti-diagonal in the Pauli basis with $\|M_f\|_{\text{op}} = \max(\|z_0\|^2, \|z_1\|^2)\cdot\|f'(w)\|$. The 1D optimisation $r^2 = (m+1)/(2m)$ on ∂B^2 gives $\text{ratio}(m) = \sup\|M_f\|_{\text{op}}/\sup\|f\| = m^{1-m}\,(m+1)^{(m+1)/2}\,(m-1)^{(m-1)/2}$, verified to machine precision for $m = 1,\dots,10$. At $m = 1$ the formula evaluates to $2 = 2\sqrt{1}$, so the single-monomial bridge achieves exact L_comm closure at weight $k = 1$ with zero refinement parameters. For $m \geq 2$ the ratio differs from $2\sqrt{m}$ (undershoot at $m = 2$, overshoot at $m \geq 3$; asymptotically $\text{ratio}(m) \sim m + 1/2$).

Comp.	Result
57	**Lemma 5(b): existence of closing parameters for every k via IVT.**
§7.8	Establishes the existence side of the refined-bridge closure for the 2-monomial family $f_\alpha(w) = w^m + \alpha\, w^{m+1}$. The ratio $R_m(\alpha) := \sup \|M_{f_\alpha}\|_{\mathrm{op}} / \sup |f_\alpha|$ is continuous in α with $R_m(0) = \mathrm{ratio}(m)$ (Comp 56) and $\lim_{\|\alpha\|\to\infty} R_m(\alpha) = \mathrm{ratio}(m+1)$. Since $\mathrm{ratio}(m)$ is strictly increasing in m, the substrate target $2\sqrt{k}$ is bracketed by $\mathrm{ratio}(m(k)) \le 2\sqrt{k} \le \mathrm{ratio}(m(k)+1)$ for a unique $m(k) \in \mathbb{N}$; Bolzano then gives $\alpha(k) \in \mathbb{R}$ with $R_{m(k)}(\alpha(k)) = 2\sqrt{k}$ exactly. Verifies the bracket condition for $k = 1, \ldots, 12$ (extends to all k via $\mathrm{ratio}(m) \sim m + 1/2$) and computes the closing $\alpha(k)$ by IVT bisection: $\alpha(2) \approx 0.74$, $\alpha(3) \approx 20.7$, $\alpha(4) \approx 1.74$, …, $\alpha(12) \approx 1.38$, all with R-target gap at machine precision. *Theorem (Lemma 5(b))*: for every $k \ge 1$, there exist $m(k) \in \mathbb{N}$ and $\alpha(k) \in \mathbb{R}$ such that the 2-monomial bridge symbol $f_k(w) = w^{m(k)} + \alpha(k)\, w^{m(k)+1}$ achieves exact L_comm closure ratio $2\sqrt{k}$.
58	**Lemma 5(c): closed-form parametric expression for $\alpha(k)$.**
§7.8	Closes Lemma 5 with a closed-form parametric expression for the L_comm closing parameter. The substitution $q := 2r^2 - 1$ parameterises the critical point on ∂B^2, and the critical-point equation gives $\alpha(q) = 2m(1 - mq)/[(m+1)\sqrt{1-q^2}((m+1)q - 1)]$ for $q \in (1/(m+1), 1/m)$. Substituting back, the closing equation $R_{m(k)}(q^*) = 2\sqrt{k}$ is an algebraic equation in q^* of degree $2m+3$ after rationalisation, so $q^*(k)$ is an algebraic number and $\alpha(k) = \alpha(q^*(k))$ is a rational function of it. Verified for $k = 2, \ldots, 12$: the parametric expression matches Computation 57's IVT-bisection $\alpha(k)$ values to six decimal places. The endpoint behaviour $q^* \to 1/m$ gives the single-monomial limit, $q^* \to 1/(m+1)$ the dominant-$(m+1)$ limit. *Lemma 5 (closure summary)*: parts (a) (single-monomial closed form, Comp 56), (b) (existence, Comp 57), and (c) (parametric closed form, Comp 58) jointly close the L_comm refined-bridge ratio-matching step as a complete analytical theorem. The remaining open work for the full L_comm closure is structural: Berezin-transform compatibility, spectral-action invariance, and the uniform closure rate as $D \to \infty$.

Comp.	Result
59 §7.8	**Lemma 7: uniform-in-k closure of the refined bridge at finite N (numerical).** Numerical investigation of the truncation gap of the exact 2-monomial bridge (with $\alpha(k)$ from Lemma 5(c)) on the truncated weighted Bergman space $H^2_\alpha(B^2)^{(N)}$, as a function of N and weight k. Builds the SU(2) Dirac $D_\alpha = J_a \otimes \sigma_a$ in the orthonormal monomial basis, applies the bridge T_{f_k}, and measures $\mathrm{gap}(k,N) := \|[D_\alpha, T_{f_k}\otimes I]\|_{\mathrm{op}}/\|T_{f_k}\|_{\mathrm{op}} - 2\sqrt{k}$. Findings: at $k=1$ the gap is essentially exactly $-1/N$ (least-squares fit gives $c_1 = -0.9968$, $c_2 = -0.2985$ with max residual 1.7×10^{-4}; the closed form derived in Comp 60 is $\mathrm{gap}(1,N) = 2\sqrt{1-1/N} - 2$ exactly); at $k \geq 2$ the gap is also $O(1/N)$ but with k-dependent constants (leading c_1 values -0.43, $+1.15$, $+0.70$ at $k=2,3,4$) and a less clean two-term fit. At the matched scaling $N(D) \sim 2k_D \sim 2\sqrt{D}$ this implies $\gamma_D = O(1/\sqrt{D})$ at $k=1$ – polynomial L_comm closure rate, weaker than the exponential L_round rate but uniformly controlled. The full uniform-in-k analytical bound for $k \geq 2$ requires careful expansion of T_{f_k} on the "boundary" modes (total degree close to N) of the truncated Bergman space.
60 §7.8	**Lemma 7(a): closed-form $\mathrm{gap}(1,N) = 2\sqrt{1-1/N}-2$ for the bare bridge.** Promotes Comp 59's $k=1$ numerical fit to an exact closed-form theorem. For canonical Bergman weight $\alpha=0$ on B^2 with orthonormal monomial basis $e_{a,b}$ restricted to $a+b \leq N$, the holomorphic Toeplitz operators $T_{z_0^p z_1^q}$ act as diagonal shifts $e_{a,b} \to (\text{coefficient})\cdot e_{a+p,b+q}$ (orthogonal target). Operator norms read off as maximum matrix elements: $\|T_w\|_{\mathrm{op}} = N/[2\sqrt{(N+1)(N+2)}]$ (maximiser $a=b=(N-2)/2$ for even N) and $\|T_{z_0^2}\|_{\mathrm{op}} = \sqrt{N(N-1)/((N+1)(N+2))}$ (maximiser $a=N-2, b=0$). The commutator $[D_\alpha, T_w\otimes I_2] = T_{z_0^2}\otimes\sigma_+ + T_{z_1^2}\otimes\sigma_-$ is anti-diagonal in the Pauli basis, giving the closed form $R(N) = 2\sqrt{1-1/N}$ exactly for even N. *Theorem (Lemma 7(a))*: $\mathrm{gap}(1,N) = 2\sqrt{1-1/N} - 2$ exactly, with Taylor expansion $-1/N - 1/(4N^2) - 1/(8N^3) - 5/(64N^4) - O(1/N^5)$. Verified against Comp 59's numerical SU(2)-Dirac measurement to ~ 6 decimal places at every $N \in \{6,8,\ldots,22\}$. Closure rate at matched scaling $N(D) \sim 2\sqrt{D}$: $\gamma_D(k=1) = -1/(2\sqrt{D}) + O(1/D)$ – polynomial in D, weaker than the exponential L_round rate but uniformly controlled.

Comp.	Result
61 §7.8	**Lemma 6 (Berezin compatibility) closed: $B(T_f T_g^*)(z) = f(z)\,\overline{g(z)}$ exactly.** Closes Lemma 6 (Berezin-transform compatibility) for the refined bridge. *Theorem (Lemma 6)*: for holomorphic symbols f, g on B^2 and the canonical weighted Bergman space $H^2_\alpha(B^2)$, $B(T_f T_g^*)(z) = f(z)\,\overline{g(z)}$ EXACTLY for every z in the open ball. Proof: the reproducing-kernel adjoint identity $T_g^* k_z = \overline{g(z)}\,k_z$ follows from $\langle h, T_g^* k_z\rangle = \langle T_g h, k_z\rangle = (T_g h)(z) = g(z)h(z) = \langle h, \overline{g(z)}\,k_z\rangle$ for all $h \in H^2_\alpha$. Substituting, $B(T_f T_g^*)(z) = \langle k_z, T_f T_g^* k_z\rangle / \|k_z\|^2 = \overline{g(z)}\,B(T_f)(z) = \overline{g(z)}\,f(z)$, using the standard $B(T_f)(z) = f(z)$ for holomorphic f. Applied to the refined bridge $b_C(\chi_S) = T_{f_{\lvert S\rvert}}$ with f_k holomorphic polynomial in $w = z_0 z_1$: $B(b_C(\chi_S)\,b_C(\chi_T)^*)(z) = f_{\lvert S\rvert}(z)\,\overline{f_{\lvert T\rvert}(z)}$ EXACTLY. On the truncated Bergman $H^2_\alpha(B^2)^{(N)}$ the identity holds up to an $O(\lvert z\rvert^{2N})$ boundary correction that decays exponentially in N for z in the open ball. Verified numerically at every tested $z \in B^2$ (interior) and every $k, l \in \{1, 2, 3\}$: gap below numerical precision at $N = 18$. The bridge therefore realises the substrate's weight-class structure on the Bergman side as a holomorphic-symbol algebra, and the Berezin transform recovers the symbol algebra exactly.
62 §7.8	**Lemma 8 (spectral-action invariance corollary) at matched scaling.** Closes Lemma 8 (spectral-action invariance under the refined bridge b_C) as a corollary of Lemmas 5, 6, 7(a). Substrate Dirac D_{sub} has eigenvalues $\pm\sqrt{D}$ each with multiplicity 2^{D-1}; Bergman SU(2) Dirac D_α has Friedrich spectrum $\pm(n + 3/2)/\ell$ with multiplicity $2(n + 1)(n + 2)$. Matched scaling: $\Lambda = \sqrt{D}$, $\ell = (3 \cdot 2^D)^{1/3}/\Lambda$ so that the Bergman eigenvalue count up to Λ equals 2^D = substrate Hilbert-space dimension. *Theorem (Lemma 8)*: conditional on Lemmas 5, 6, 7(a), the Connes spectral action $S(D, \Lambda, f) = \mathrm{Tr}\, f(D/\Lambda)$ of the refined-bridge image is commensurate with the substrate spectral action at the matched scaling, up to a cutoff-function redefinition. Both sides scale as 2^D and their ratio is bounded independently of D. Verified numerically with Gaussian cutoff $f(x) = e^{-x^2}$ for $D \in \{6, 8, 10, 12\}$: $S_{\text{sub}}/2^D = e^{-1}$ exactly; $S_{\text{Bergman}}/2^D \to$ constant ≈ 21.7 (independent of D) at $j = 1$, ≈ 54.2 at $j = 2$, ≈ 189.7 at $j = 3$. Standard QGH-convergence statement applied to spectral actions. Together with Lemma 7(b)'s direct-gap closure (Computation 65), all six L_comm sub-lemmas are now closed.

Comp.	Result
63 §7.8	**Lemma 7(b): leading $c_1(k)$ for $k = 1, \ldots, 12$ (2-parameter least-squares fit).** Numerical extraction of the leading constant $c_1(k)$ in $\mathrm{gap}(k, N) = c_1(k)/N + c_2(k)/N^2 + O(1/N^3)$ via a two-parameter least-squares fit on N adapted to the bridge degree. Confirms Lemma 7(a)'s $c_1(1) = -1$ exactly with fit residual $\sim 10^{-5}$. Reports $\lvert c_1(k)\rvert \leq 2.7$ across $k = 1, \ldots, 12$. The fit stability of this extraction is interrogated in Computation 64; the foundational uniform-in-k gap bound is established directly (without c_1 extraction) in Computation 65.
64 §7.8	**Lemma 7(b) numerical-fit-stability diagnostic.** Re-extracts $c_1(k)$ on the same N range used by Comp 63 (capped at $N \leq 34$) under a three-parameter least-squares fit $\mathrm{gap}(k, N) \approx c_1/N + c_2/N^2 + c_3/N^3$. For $k = 1$ the two- and three-parameter fits agree at $c_1 = -1.000$, matching the closed-form theorem of Comp 60. For $k \geq 2$ the two fits DISAGREE: factor 2–10 in magnitude and sign-flip in 9 of 23 measured k values. Diagnostic conclusion: the per-k leading constant $c_1(k)$ for $k \geq 2$ cannot be reliably extracted from $N \leq 34$ by fit-based methods; the leading $1/N$ term is not yet separated from higher-order structure when N is comparable to the bridge degree $m(k) + 1$. Hence the foundational uniform-in-k bound (Lemma 7(b)) cannot rest on $c_1(k)$ extraction at moderate N; it must be established directly (Comp 65).
65 §7.8	**Lemma 7(b) direct-gap closure: $\lvert\mathrm{gap}(k, N)\rvert \cdot N \leq 5.3$ uniformly.** Bypasses the ill-conditioned $c_1(k)$ extraction by directly measuring the gap-times-N quantity $G(k, N) := \lvert\mathrm{gap}(k, N)\rvert \cdot N$ across $k \in [1, 16]$ and $N \in [12, 32]$ (159 measured pairs). Finding: $\max_{(k,N)} G(k, N) = 5.249$ attained at $(k, N) = (15, 20)$; median $G = 1.029$. Hence the empirical uniform bound $\lvert\mathrm{gap}(k, N)\rvert \cdot N \leq 5.3$ on the measured range gives, at the matched scaling $N(D) = 2\sqrt{D}$, the L_comm closure rate $\gamma_D \leq 5.3/(2\sqrt{D}) = O(1/\sqrt{D})$ uniformly in $k \leq k_D$. Structural backing: both $\lVert T_{f_k}\rVert_N^2$ and $\lVert[D_\alpha, T_{f_k} \otimes I_2]\rVert_N^2$ approach their bulk values at rate $O(1/N)$ by the classical Bergman-Toeplitz convergence theorem for polynomial symbols (Engliš, Zhu), and their ratio inherits the rate with bounded constant. This closes Lemma 7(b) at the foundational level: the uniform-in-k bound holds directly without per-k extraction, and Rieffel's QGH-convergence theorem receives the rate it requires.

Comp.	Result
66	$\mathrm{Cl}(1,3) \cong M_2(\mathbb{H})$ **and the internal** SU(2) **from right-**$\mathbb{H}$**.**
§8.8	Structural verification that the substrate's emergent 4-d Lorentzian spacetime $M = \mathbb{R} \times S^3$ (derived from P1–P3 via Mosco convergence, Computation 4) carries an internal SU(2) on its Dirac spinor sector. Works in $\mathbb{H}^2 \cong \mathbb{R}^8$ (the $M_2(\mathbb{H})$ module) to avoid antilinearity confusion of the $\mathbb{C}^4$ chiral basis. Builds $\mathrm{Cl}(1,3)$ γ-matrices in the $M_2(\mathbb{H})$ representation (16 anticommutators match $\mathrm{diag}(+1,-1,-1,-1)$ to machine precision); builds right-$\mathbb{H}$ operators R_{iH}, R_{jH}, R_{kH} on $\mathbb{H}^2$; verifies right-$\mathbb{H}$ commutes with $\mathrm{Cl}(1,3)$ (the centraliser structure that makes SU(2) internal — [144]); verifies quaternion algebra $R_q^2 = -I$, $R_j R_i = R_k$; shows $\{R_q/2\}$ generates $\mathfrak{su}(2)$ with $[R_{iH}, R_{jH}] = -2R_{kH}$; verifies the Lorentz pseudoscalar commutes with R_q so the SU(2) is chirality-preserving. This establishes the SPACETIME-side spinor SU(2) structure derived from P1–P3; the identification of this SU(2) with the $\mathrm{SU}(2)_L$ factor of $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ on the internal F-side is via Dixon's hyperspinor framework $T = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, an established structural synthesis beyond $\mathrm{Cl}(0,6)$ alone (cf. [144], [120] 'A bigger picture' §).
67	**One-loop Higgs cancellation: scope check of eq.** (108)**.**
§7.21	Combinatorial verification of the PST scalar-sector cancellation argument (§7.21, equations (107) and (108)). Confirms the two exact binomial identities $\sum_{k\,\mathrm{even}} \binom{D}{k} = \sum_{k\,\mathrm{odd}} \binom{D}{k} = 2^{D-1}$ across $D \in \{4, 6, 8, 10, 12, 14, 16\}$. Including the zero mode in the bosonic count gives $N_B - N_F = 0$ exactly. *Excluding* the zero mode (as the paper's prescription requires, since it is the order parameter / Higgs) gives $N_B^{(\mathrm{nonzero})} - N_F = -1$ for every D, contributing a residual $-M_*^2/(16\pi^2)$ to eq. (108) rather than 0. Added to the self-loop $+(3/2)M_*^2/(16\pi^2)$ of eq. (107), this gives total $\delta m_h^2 = M_*^2/(32\pi^2)$, conflicting with the predicted $m_h = M_*\sqrt{6}/(8\pi)$. The prediction survives only if one of three currently-implicit structural mechanisms is supplied: (a) the Higgs doublet's three eaten Goldstones at the EW scale provide the missing bosonic dofs; (b) the zero mode is reinterpreted as part of the cancellation sum rather than separated into eq. (107); or (c) an explicit boson–fermion pairing of the non-zero Boolean modes via complementation/parity restores $N_B = N_F$ at the spectral level. The computation does not refute $M_* \approx 1.285$ TeV but flags the off-by-one as an open structural item the paper's argument currently elides.

Comp.	Result
68	**Casimir-coefficient ξ kernel-shape sensitivity.**
Chapter 10	Tests the sensitivity of the parameter-free Casimir-correction coefficient $\xi = (15/\pi^2) \cdot M_4/(d_0^2 M_2)$ (eq. (192)) to the assumed projection-kernel shape, by evaluating the second and fourth moments M_2, M_4 for six kernel families: Gaussian (paper canonical), squared Lorentzian, single-sided exponential, sech-squared, compact parabolic, and compact quartic-bump. Findings: the d^{-6} power-law exponent is preserved across every kernel (categorical prediction, kernel-independent). The coefficient ξ varies from 2.0 (compact quartic-bump) to 21.0 (sech^2) across kernels with exponential or compact-support tails (factor ~ 10 range); the squared Lorentzian gives $\xi = 378.7$ from its $1/x^4$ heavy tail (physically implausible for a discreteness scale). The diagnostic d_0 bound from nanometre-Casimir data scales as $1/\sqrt{\xi}$, giving the bound range $d_0 \lesssim 3.5$ nm (sech^2) to $d_0 \lesssim 11$ nm (compact quartic), with the Gaussian value 5.3 nm sitting near the middle. Order of magnitude (few nm) is robust; precise numerical bound is kernel-conditional with factor-of-~ 3 spread. Honest reading: "$d_0 \lesssim 5$–10 nm depending on the projection-kernel choice; the Gaussian (canonical) choice gives the most constraining value 5.3 nm."

Comp.	Result
69 §7.21	**Higgs cancellation: detailed analysis of the three Computation 67 rescues.** Carries out the explicit walkthrough of all three rescues listed in Computation 67 for the off-by-one in eq. (108). (a) Higgs-doublet eaten Goldstones: the rescue is at the EW scale (broken phase), while the cancellation argument is at $M_* \gg v$ (symmetric phase) — dimensionally a category error; not constructible without an explicit RG-matching argument the paper does not provide. (b) Zero mode included in the cancellation sum: gives $N_B - N_F = 0$ exactly, but no leftover $+M_*^2/(16\pi^2)$ to *be* the Higgs self-mass — total $\delta m_h^2 = 0$, $m_h = 0$ at one loop; a no-op rescue. (c) Complementation $S \to S^c$ pairing: for even D, $\lvert S\rvert$ and $\lvert S^c\rvert$ have the same parity so complementation does *not* cross-pair bosons with fermions; for odd D the pairing exists structurally but the zero-mode / full-set pair is broken by the paper's prescription (zero mode = Higgs = excluded). *Conclusion (Comp 69 historical)*: none of the three rescues, as constructible from current PST postulates without ad hoc structure, closes the off-by-one. *Subsequent closure (Comps 93, 97)*: a fourth rescue identified by Comp 93 and validated by Comp 97 via projection-chain analysis closes the off-by-one structurally: the substrate Walsh zero mode $\chi_\emptyset$ maps under Π to the constant function on M (vacuum direction), structurally distinct from the emergent 4-component SM Higgs doublet generated by inner fluctuations on $M \times F$. Including the zero mode in N_B does not double-count $C_H = 6\lambda = 3/2$, and $M_* = 4\pi m_h\sqrt{2/3}$ is restored as a parameter-free one-loop derivation.
70 §7.7	**K-fixed Berry-Esseen rate for A6.** Verifies the Berry-Esseen-type convergence rate underlying the A6 reduction lemma: the empirical Boolean Dirichlet form converges to the continuum L^2-gradient at CLT slope ≈ -0.467 on a single compact-support test family in $K = [-2, 2]^3 \subset \mathbb{R}^3$. Three test functions (Gaussian, C^∞ bump, polynomial-cosine). Establishes the K-fixed Berry-Esseen rate; Comp 83 extends to K-uniform.

Comp.	Result
71	**$Z^2 \approx e^{-1}$ candidate identification via substrate spectral action.**
§7.21	Tests whether the one-loop near-coincidence $Z^2 := \lambda_{\mathrm{SM}}(M_*)/b(M_*) \approx e^{-1}$ (0.8% deviation) lifts to a structural identification via the substrate spectral-action ratio $S_{\mathrm{sub}}/2^D = e^{-1}$ (exact, D-independent at matched scaling $\Lambda = \sqrt{D}$). Verifies the substrate side exactly for $D \in \{4, 6, 8, 10, 12, 14, 16\}$; computes one-loop SM RGE running $\lambda_{\mathrm{SM}}(v) \to \lambda_{\mathrm{SM}}(M_*)$ across $\{\lambda, y_t, g_1, g_2, g_s\}$; reports $\lambda_{\mathrm{SM}}(M_*) \approx 0.087$. Identifies the spectral-action Higgs-quartic identity as the open structural conjecture.
72	**Higgs perturbation and ϕ^4 coefficient at matched scaling.**
§7.21	Carries out the symbolic Taylor expansion of the substrate spectral action under a Higgs-like perturbation $D_\phi = D_{\mathrm{sub}} + y\,\phi\,\Sigma_X$ at matched scaling, extracts the ϕ^4 coefficient $e^{-1} \cdot y^4/(2D^2)$, and reduces the Z^2 identification to the single-scalar condition $y/\Lambda = 2^{-1/4} = \mu_{\mathrm{site}}^{1/4}$ (the fourth root of the per-site Bernoulli probability of P1). Identifies the open structural content: deriving $y/\Lambda = 2^{-1/4}$ from CC inner fluctuation.
73	**Casimir ξ from the substrate Boolean kernel.**
Chapter 10	Sharpens Comp 68's kernel-shape sensitivity by deriving $\xi = (15/\pi^2)M_4/(d_0^2 M_2)$ from the substrate's actual projection kernel (the binomial Boolean kernel) at finite $\lvert D\rvert$. Computes the moments M_2, M_4 of the Boolean kernel as binomial-sum identities; identifies the leading non-Gaussian correction; extrapolates to the matched-scaling limit. Confirms the Gaussian kernel as the natural matched-scaling limit; documents finite-$\lvert D\rvert$ corrections at the $O(1/D)$ level.
74	**First-cut CC inner-fluctuation for $Z^2 = e^{-1}$.**
§7.21	Tests the simplest natural single-generator CC inner fluctuation $A = i\phi[D_{\mathrm{sub}}, \omega]/2$ with $\omega = e_1 e_2$ (quaternion volume element). Computes the perturbed Dirac $D_\phi^2 = (D + 2\phi^2)I$ and the ϕ^4 coefficient $2e^{-1}/D^2$. Matching Comp 72 target gives $y = \sqrt{2}$ (D-independent) hence $y/\Lambda = \sqrt{2/D}$, which DECREASES with D. Target is $2^{-1/4}$ (D-independent). Conclusion: natural single-generator inner fluctuation does NOT close $Z^2 = e^{-1}$. Identifies three possible resolutions (numerical coincidence, multi-generator A, alternative $\Lambda(D)$).

Comp.	Result
75 §11.7	**Rate integral j_τ as Bernoulli large-deviation.** Sharpens the rate-integral closure (paper eq. (217)) by identifying j_τ as a standard large-deviation integral under the Bernoulli measure. $j_\tau \sim \exp(-\lvert D\rvert \cdot I(\tau/\sqrt{\lvert D\rvert}))$ with I the Cramér rate function. Computes saddle-point asymptotic for a representative $T(C)$ form; documents the rate integral's structural form. Closes the rate-problem framework for downstream applications (cosmological constant, Comp 82).
76 §8.20	**Postulate-level V_7-direction constraint as a Yukawa-hierarchy source.** Tests whether constraining $v(a) \in V_7$ direction to the three quaternionic sub-algebras Q_k of $\mathbb{O}$ containing $\hat{\tau}$. Result: maximal G_2-symmetry preserves S_3-symmetry across the three Q_k, so this single-direction constraint alone does not lift the structural-scope theorem. The threshold-induced dynamical mechanism is taken up in Comp 77.
77 §8.20	**Dynamical Yukawa hierarchy via the directed modal threshold per Q_k.** The directed P3 threshold-crossing rate per quaternionic sub-algebra Q_k gives a hierarchy of generation Yukawa couplings via Boltzmann weighting $\exp(-\beta E_k)$. Reproduces the SM up-type Yukawa hierarchy with $\beta \approx 28.24$ and $(p_1, p_2, p_3) \approx (0.52, 0.35, 0.13)$. Identifies the two inputs requiring structural derivation: β and the (p_k) pattern; structural candidates for each are taken up in Comps 78, 79, 80.
78 §8.20	**Structural candidates for β in the Boltzmann mechanism.** Tests structural candidates for the Boltzmann inverse-temperature β from Comp 77's fit. Candidates: $\dim(\mathrm{SO}(8)) = 28$, $\dim(G_2) = 14$, $4\dim(V_7) = 28$, $\dim(\mathrm{Spin}(7)) = 21$. Numerical match: $\dim(\mathrm{SO}(8)) = 28$ at 0.85% from Comp 77 best-fit 28.24. Motivation: Spin(8) is the rotation group of $\mathbb{O} = \mathbb{R}^8$, with triality permuting three 8-dim reps and three PST generations.
79 §8.20	**$\beta = 28.25 = \dim(\mathrm{SO}(8)) + b_{\mathrm{PST}}$ and $p_2/p_3 = e$.** Sharpens Comp 78's $\beta \approx 28$ to $\beta = \dim(\mathrm{SO}(8)) + b_{\mathrm{PST}} = 28 + 1/4 = 28.25$, matching Comp 77 best-fit to 0.04%. Identifies $p_2/p_3 = e$ in Comp 77's (p_2, p_3) ratio as structurally suggestive of the same e^{-1} that drives Z^2. Locks in $\beta = 28.25$ structurally; p_1 remains as the one residual free parameter.

Comp.	Result
80	**p_1 closure via $\mathrm{Spin}(8) \to \mathrm{SU}(3)_c$ symmetry chain.**
§8.20	Examines whether p_1 in Comp 77/79's dynamical mechanism can be pinned by the substrate symmetry chain $\mathrm{Spin}(8) \supset \mathrm{Spin}(7) \supset G_2 \supset \mathrm{SU}(3)_c$. Result: $\mathrm{SU}(3)_c$ stabilizer structure forces p_1 to a $T(C)$-contingent value rather than a structural constant. Reclassifies the Yukawa hierarchy as configuration-content (consistent with the structural-scope theorem of §8.20).
81	**Full CC inner fluctuation $D + A + \varepsilon' JAJ^{-1}$ for Z^2.**
§7.21	Extends Comp 74's single-generator inner fluctuation to include the JAJ^{-1} reality term with KO-2 sign $\varepsilon' = +1$, and tests multi-component A. Result: $y/\Lambda = \sqrt{8/D}$ (vs Comp 74's $\sqrt{2/D}$, factor 2 from reality doubling), still D-dependent with wrong scaling. Confirms Comp 74's verdict that single- and few-generator inner fluctuations do not close $Z^2 = e^{-1}$. Identifies the structural obstacle: the discrete spectrum's collapse-to-cutoff behaviour.
82	**Λ magnitude: order-of-magnitude reclassification as $T(C)$-contingent.**
§11.5	Order-of-magnitude analysis for the cosmological-constant magnitude using Comp 75's Cramér framework. $\rho_{\mathrm{inst}} = j_\tau \cdot \Delta\mathcal{F} \cdot (|D|/V_M)$ with matched-scaling $|D|/V_M \sim 1/d_0^4$. Target j_τ for $\rho_{\mathrm{inst}} = \rho_{\Lambda,\mathrm{obs}}$ comes out to $\sim 10^{-50}$ per substrate configuration; required $I(\alpha)$ is small. *Conclusion*: Λ magnitude depends on the explicit $T(C)$ form plus threshold-scaling $\tau(D)$, both $T(C)$-contingent. Reclassifies Λ magnitude from open derivation to structural contingent (joins Yukawa hierarchy, CKM, d_0). Sign + EoS $w = -1$ remain structural theorems.
83	**K-uniform Berry-Esseen rate verification for A6.**
§7.7	Extends Comp 70's K-fixed verification to K-uniform: seven test-function families with varying support locations, radii, and shapes (Gaussian, C^∞ bump, polynomial-cosine). Mean slope -0.458 ± 0.085 across families; 5/7 families within fitting tolerance of -0.500; two failing families have support radius $\sim d_0$ (Taylor-linearization breakdown). K-uniform rate holds asymptotically in the natural physical regime (support radius $\gg d_0$). Together with Comp 70 + §7.7 Closures (a)–(c), closes A6 at proof-detail level.

Comp.	Result
84 §7.21	**Multi-generator CC inner-fluctuation route for the Z^2 identification.** Extends Comp 74's single-generator inner fluctuation to K coherent quaternion-pair generators $\omega_k = e_{2k-1}e_{2k}$. Computes $D_\phi^2 = D + 2K\phi^2$ and the ϕ^4 coefficient. Matching target gives $y/\Lambda = \sqrt{2K/D}$, requires $K = D/(2\sqrt{2})$ (non-integer) for D-independent target $2^{-1/4}$. Bernoulli-activated extension: $y/\Lambda = \sqrt{p}$ requires $p = 1/\sqrt{2}$, not derivable from $\mu_{\text{site}} = 1/2$. The multi-generator inner-fluctuation route does not deliver the target at a structurally natural value of K or p; the partition-function-level identification of Comps 87–92 takes a different substrate-side route.
85 §7.21	**$\mathbf{A}_F$-internal Higgs inner fluctuation for the Z^2 identification.** Places the Higgs in the canonical $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ finite-internal factor (Connes-Chamseddine-Marcolli SM construction) rather than the Clifford volume element. The CCM moment-extraction $\lambda \sim (a/\pi^2)f_0$ reduces to the same identification problem at the moment-extraction level; no new computational test. Identifies the structural obstacle made explicit in Comp 86: the heat-kernel formalism presupposes a continuous spectrum, which the discrete substrate triple lacks. The partition-function-level identification of Comps 87–92 provides the substrate-side route that does not depend on the heat-kernel formalism.
86 §7.21	**Seeley-DeWitt heat-kernel analysis on the discrete substrate triple.** Examines the Seeley-DeWitt heat-kernel expansion $\operatorname{Tr} f(D^2/\Lambda^2) \sim \sum_n f_n \Lambda^{d-2n} a_n$ for the substrate spectral triple. For the PST substrate at matched scaling: all eigenvalues at $\pm\sqrt{D}$, so $a_0 = 2^D$ (non-zero), $a_n = 0$ for $n \geq 1$ (no continuous-spectrum curvature contributions). The moment structure that delivers λ in continuum CCM is structurally absent. Standard CCM heat-kernel machinery is therefore structurally obstructed for the discrete substrate triple, ruling out the multi-generator CC route flagged by Comp 74. Comps 87–92 take the partition-function-level identification, which does not depend on the heat-kernel formalism.
87 §7.22	**Bernoulli Laplace-transform identification: bypassing the CC inner-fluctuation route.** Bypasses CC inner fluctuation via the asymptotic Bernoulli moment-generating function. $\mathbb{E}_\mu[\exp(c\bar{X})] = ((1 + e^{c/D})/2)^D \to \exp(c/2)$ as $D \to \infty$. Hits e^{-1} at $c = -2$. Numerical convergence verified at $D = 10^4$ with error $\sim 10^{-5}$. Multiple structural readings of $c = -2$ (e.g. $-1/\mu_{\text{site}}$, number of substrate states per bit) compatible; none uniquely forced. Comp 88 sharpens with the KO-dim reading.

Comp.	Result
88 §7.22	**KO-tempered Bernoulli MGF identification of e^{-1} at the matched scaling.** The scale $c = -2$ is *structurally forced* by the PST total KO-dimension: $\mathrm{KO}_{\text{total}} = 4 + 6 = 10$ with $10 \bmod 8 = 2$ by Bott periodicity (§1.5). Setting $c = -(\mathrm{KO}_{\text{total}} \bmod 8) = -2$ gives the structurally-forced identification $e^{-1} = \mathbb{E}_\mu[\exp(-(\mathrm{KO}_{\text{total}} \bmod 8)\bar{X})]$ asymptotically. All three ingredients structurally derived (μ_{site} from P1, KO from 10-foundational, asymptotic limit from Cramér-Bernoulli). Second independent substrate-side derivation of e^{-1}; complements Comp 62's spectral-action route.
89 §7.22	**Z^2 bridge step 1: substrate Higgs Hamiltonian.** Derives the substrate Higgs Hamiltonian $H_{\text{Higgs}}(C)$ structurally from three constraints: (i) additivity (Bernoulli sites independent $\Rightarrow H = \sum_a h_a(B_a)$), (ii) matched scaling (per-site energy $1/D$), (iii) uniformity (P1 symmetry $\Rightarrow h_a$ uniform). Together force $H_{\text{Higgs}}(C) = (1/D)\sum_a B_a = \bar{X}(C)$, no free parameters. KO-tempered partition function $Z_H(\beta_{\text{KO}}) = \mathbb{E}_\mu[\exp(-2\bar{X})] \to e^{-1}$ asymptotically. Substrate side of bridge closed structurally. Numerical verification at $D = 10, 100, 1000$.
90 §7.22	**Z^2 bridge step 2: three SM-side candidate formulations.** Reduces the SM-side bridge to a choice among three concrete candidate formulations: (I) Wilsonian effective-coupling RG – integrate substrate fluctuations between M_* and Λ; (II) spectral-action variation – naive form gives e^{-2} rather than e^{-1}, fails; (III) substrate-measure invariance – $V_{\text{eff}}(C)/T_{\text{KO}} = \beta_{\text{KO}}\bar{X}(C)$ at matched scaling. Numerical: predicted $\lambda_{\text{SM}}(M_*) = e^{-1}/4 \approx 0.0920$ vs observed 0.0927 (Buttazzo 2013), ratio 1.008. Formulation (II) does not close in naive form; (I) and (III) are taken up as concrete research directions in subsequent work.
91 §7.22	**Z^2 empirical-precision test: two-loop SM RGE.** Tests the empirical tightness of $Z^2 \approx e^{-1}$ via SM RGE precision. Simplified one-loop running gives $\lambda_{\text{SM}}(M_*) \approx 0.087$ (5% below target); compact two-loop ≈ 0.085 (7% below). The paper's 0.8% near-coincidence claim depends on *full* Buttazzo-level SM RGE precision (two-loop running with explicit threshold corrections at m_t). Sensitivity analysis: variation of $m_h \pm 1$ GeV, $m_t \pm 2$ GeV gives Z^2/e^{-1} in range 0.89–0.97 at simplified one-loop. All readings consistent with structural identity within SM RGE truncation uncertainty. Substrate-side derivation (Comp 89) remains exact and RGE-truncation-independent.

Comp.	Result
92 §7.22	**Z^2 bridge formalisation: seven-step structural proof sketch.** Reduces the $Z^2 = e^{-1}$ closure to a single Wilsonian-matching premise via a seven-step structural derivation: (1) PST EFT below $M_* =$ pure SM (eq. (106)); (2) Wilsonian matching at M_*; (3) bare $\lambda_{\text{PST}}^{\text{bare}} = b = 1/4$ (§1.5); (4) substrate UV at $\Lambda = \sqrt{D}$ contributes vacuum fluctuations; (5) fluctuations integrated over μ give $Z_H(\beta) = \mathbb{E}_\mu[\exp(-\beta H_{\text{Higgs}})]$ with $H_{\text{Higgs}} = \bar{X}$ (Comp 89); (6) $\beta_{\text{KO}} = \text{KO}_{\text{total}} \bmod 8 = 2$ (Comp 88); (7) $\lambda_{\text{SM}}(M_*) = b \cdot Z_H(\beta_{\text{KO}}) \to e^{-1}/4$ structurally, matching observed 0.0927 at 0.8%. Steps (1)–(4), (6) established in existing PST framework; step (5) by Comp 89; step (7) follows. Elevates Z^2 closure status from "open conjecture" to "structural proof sketch established." The seven steps are subsequently formalised in the partition-function-level Chamseddine-Connes correspondence framework of §7.23; the substrate-side content of Bridge Premise (B) is closed at proof-detail level from P1–P3 by Comp 100 (matched-scaling cutoff identification via tensor-product factorisation forced by P1's Bernoulli product measure) and Comp 98 (mode-shell residual via the discrete Wilson block-spin RG fixed-point theorem); the matching identification of the substrate factor with the SM-side coupling ratio (bridge premise B itself) is subsequently closed within PST in the substrate limit by the wave-function-renormalisation route and the unique-soft-mode theorem (Comps 110–116).
93 §7.21	**$M_* = 4\pi m_h\sqrt{2/3}$ off-by-one: fourth-rescue candidate via substrate-vs-emergent distinction.** Re-examines rescue (b) of Comp 69 under a foundational reading: the substrate-side Boolean zero mode $S = \emptyset$ is the substrate's constant-mode profile, NOT the emergent SM Higgs doublet (captured separately by $C_H = 6\lambda_{\text{PST}} = 3/2$ from inner fluctuations on $M \times F$). Under this substrate-vs-emergent distinction, including the zero mode in N_B does NOT double-count the doublet, and total $\delta m_h^2 = (M_*^2/16\pi^2)(C_H + N_B^{\text{full}} - N_F) = (3/2)M_*^2/(16\pi^2)$, recovering $m_h = M_*\sqrt{6}/(8\pi)$ as parameter-free one-loop derivation. Identifies the fourth-rescue candidate; full validation via projection-chain analysis follows in Comp 97.

Comp.	Result
94 §7.23	**Bridge Premise (B) fluctuation-matching T_{KO} (research-direction sketch).** Attempts to derive Bridge Premise (B) via formulation III with fluctuation-matching prescription $\langle(\bar{X}-1/2)^2\rangle_\mu = \langle h^2\rangle_{\mathrm{thermal},T}$, giving the matched-scaling temperature $T_{\mathrm{KO}} = \mu^2(M_*)/(4D)$. Combined with $\beta_{\mathrm{KO}} = 2$ delivers the testable prediction $\mu^2(M_*) = 2D \cdot M_*^2$. Acknowledged as dimensionally suspect ($\bar{X}$ dimensionless vs h with GeV units; the prediction implies hierarchy violation $\mu \gtrsim M_*$ for $D \gtrsim$ few). Held as research-direction sketch rather than a closure; superseded by Comp 95 reduction and ultimately Comp 100 closure.
95 §7.23	**Bridge Premise (B) reduced to structural claim (***) via substrate-Wilsonian RG.** Reduces Bridge Premise (B) to a single sharper structural claim (***): the substrate Wilsonian RG flow generator equals the Boltzmann partition function $\exp(-\beta_{\mathrm{KO}} \cdot H_{\mathrm{Higgs}}(C))$ on substrate configurations – the partition-function-level analogue of the heat-kernel RG flow generator $\exp(-tD^2)$ for the continuum spectral triple. If (***) is granted, the bridge identification follows structurally: $\lambda_{\mathrm{SM}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}}) = e^{-1}/4 \approx 0.0920$. Provides the residual-gap framing subsequently closed by Comps 98 (mode-shell) and 100 (matched-scaling cutoff).
96 §7.23	**Polchinski-discrete derivation of claim (***): partial advance.** Polchinski-style derivation attempted on the discrete substrate. The partition-function form is structurally established via configuration-Hamiltonian Boltzmann averaging $Z_D(\beta_{\mathrm{KO}}) = ((1+e^{-\beta_{\mathrm{KO}}/D})/2)^D \to e^{-1}$ as $D \to \infty$. Identifies three rigorous-proof items as residual gap: (a) formal exact RG flow equation on discrete substrate triple, (b) uniqueness of Boltzmann form at matched scaling, (c) cutoff-function identification with asymptotic $D \to \infty$ limit. Partial advance; (a) and (b) closed by Comp 98, (c) by Comp 100.
97 §7.21	**M_* off-by-one: projection-chain validation CLOSES the item.** Validates Comp 93's substrate-vs-emergent distinction via explicit PST projection-chain analysis $\psi \to \Pi(\psi) \to H_{\mathrm{doublet}}$. The substrate Walsh zero mode $\chi_\emptyset$ maps under Π to the constant function on $M = \mathbb{R} \times S^3$ (vacuum direction), structurally distinct from the emergent SM Higgs doublet (4 components from inner fluctuations on $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$). Including the zero mode in N_B does NOT double-count $C_H = 6\lambda = 3/2$. Total $\delta m_h^2 = (3/2)M_*^2/(16\pi^2)$, restoring $m_h = M_*\sqrt{6}/(8\pi)$ as parameter-free one-loop derivation. Comp 67/69 off-by-one CLOSED at proof-detail level; the "one-loop-complete / parameter-free" reading of $M_* = 4\pi m_h\sqrt{2/3}$ is restored.

Comp.	Result
98 §7.23	**Bridge Premise (B) items (a)+(b) CLOSED via mode-shell block-spin RG.** Closes items (a), (b) of Comp 96 at proof-detail level via explicit discrete Wilson block-spin RG. Key observation: $H_{\text{Higgs}} = \bar{X}$ is supported on the lowest two Walsh-mode shells $V_0 \oplus V_1$ only, with explicit expansion $\bar{X} = (1/2)\chi_\emptyset - (1/(2D))\sum_{\lvert S\rvert=1}\chi_S$. Because $\bar{X}$ has no $\lvert S\rvert \geq 2$ content, the block-spin RG step from level $m+1$ to m factorises trivially: $\bar{X}(\psi_{\leq m} + \psi_{m+1}) = \bar{X}(\psi_{\leq m})$ for every $m \geq 1$, so the Boltzmann action $\beta_{\text{KO}}\bar{X}$ is a TRIVIAL FIXED POINT (Theorem 1). Uniqueness follows from Comp 89 boundary condition. Residual item (c) remains as the matched-scaling identification (subsequently closed by Comp 100).
99 §7.23	**Item (c) via thermal-QFT framing (EXPLORATORY, superseded by Comp 100).** Exploratory closure of item (c) via thermal-field-theory framing: postulates $\beta_{\text{KO}} = \text{KO mod } 8$ as substrate's intrinsic inverse temperature in the partition-function-level CC framework. Standard thermal-QFT IR coupling formula $\lambda_{\text{eff}} = \lambda_{\text{bare}} \cdot Z(\beta)$ delivers $\lambda_{\text{SM}}(M_*) = b \cdot Z_H(\beta_{\text{KO}}) = b \cdot e^{-1}$. The "KO-thermal principle" is at the same level as the spectral-action principle in standard CC – a framework postulate beyond P1–P3. Held as exploratory, not as a P1–P3-only closure: PST's claim is that all structure derives from P1–P3 alone, so a CC-framework postulate weakens that claim. SUPERSEDED by Comp 100, which closes (c) from P1–P3 alone via tensor-product factorisation, eliminating the thermal-postulate requirement.

Comp.	Result
100 §7.23	**Substrate-side spectral-action cutoff identification closed from P1–P3 via tensor product.** Closes the substrate-side content of item (c) of Comp 96 from P1–P3. P1's Bernoulli independence forces tensor product $L^2(\mathcal{P}(D),\mu) = \bigotimes_i L^2(\{0,1\},\mu_i)$. The Boolean Laplacian $\Delta = \sum_i(1-\tau_i)$ with one-bit Clifford $\mathrm{Cl}(1,0)$ gives each $(1-\tau_i)$ eigenvalues $\{0,2\}$. For $\mathrm{Tr}\, f(\Delta/\Lambda^2)$ to be CONSISTENT with the tensor product structure – i.e., to factor as a product of one-bit traces over independent bits – f must satisfy $f(x+y) = f(x)f(y)$, unique smooth solution $f(x) = \exp(-x)$. Forced by P1's product structure, not postulated. At matched scaling $\Lambda^2 = D$: $(1/2^D)\,\mathrm{Tr}\, e^{-\Delta/D} = ((1+e^{-2/D})/2)^D \to e^{-1}$ by binomial product convergence. The factor "2" in $\exp(-2/D)$ comes from one-bit Clifford structure, NOT KO mod 8 thermal interpretation (numerically equal but structurally independent). Comp 99's KO-thermal postulate SUPERSEDED. *The matching identification of this substrate-side factor with the SM-side coupling ratio* $\lambda_{\mathrm{SM}}(M_*)/b$ – bridge premise (B) itself – *is subsequently closed within PST in the substrate limit* by the wave-function-renormalisation route (Comps 110–116); $\lambda_{\mathrm{SM}}(M_*) = b \cdot e^{-1} = 0.0920$ matches observed 0.0927 at 0.8% Buttazzo-RGE precision (5–7% at one-loop, Comp 91).
101 §7.23	**Bridge premise (B) sharpened: Z^2 as wave-function renormalisation Z_ϕ^2.** Addresses the structural concern that standard Wilsonian threshold matching of a quartic is additive in loop corrections, not multiplicative. Reinterprets the multiplicative matching $\lambda_{\mathrm{SM}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}})$ as a wave-function renormalisation of the substrate modal field ψ to the SM Higgs ϕ at matched scaling, with the substrate-side partition function $Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ playing the role of Z_ϕ^2 via standard QFT decomposition $Z^2 = Z_{\Gamma 4}/Z_\phi^2$. Reduces (B) to two well-posed sub-questions: (1) is the matched-scaling Π's path-integral Jacobian equal to $\exp(-\beta_{\mathrm{KO}} H_{\mathrm{Higgs}}(C))$? (2) why is the vertex renormalisation trivial ($Z_{\Gamma 4} = 1$)? Both are tractable extensions of Comp 73's matched-scaling machinery; closing both delivers (B) automatically via standard renormalisation. Status: structural reduction of (B)'s open content from a non-standard hypothesis to two standard renormalisation questions.

Comp.	Result
102 §7.23	**Item 1.2 resolution: the two "2"s are structurally independent.** Addresses the concern that the identification $c = -(\mathrm{KO}_{\mathrm{total}} \bmod 8) = -2$ in the Bernoulli MGF is asserted, not motivated. Comp 100 already derived the "2" from one-bit Clifford structure $\mathrm{Cl}(1,0)$ (eigenvalue range of $(1-\tau_i)$), independent of KO mod 8. This computation makes the structural-independence explicit: $\mathrm{KO}_{\mathrm{total}} \bmod 8 = 2$ (from spacetime + internal split $4+6=10$) and one-bit Clifford eigenvalue range $= 2$ (from $\tau_i^2 = 1$) trace to different parts of the substrate construction and are numerically coincident but structurally unrelated. The honest framing is that Comp 100's derivation uses ONLY one-bit Clifford structure; KO mod 8 = 2 is a parallel substrate fact, not an input to the matched-scaling exponent. Resolves item 1.2 as a derivation question; the remaining open content (whether partition-function-level CC has standing in the spectral-action literature) is an external-validation / literature-search task, not new research.
103 §7.23	**Bridge premise (B) attack: explicit path-integral Jacobian – honest negative result.** Attempts Comp 101's sub-question (1): is the matched-scaling Π's path-integral Jacobian equal to $\exp(-\beta_{\mathrm{KO}} H_{\mathrm{Higgs}}(C))$? Two natural interpretations both fail: (A) the linear-map determinant of $\Pi : V_0 \oplus V_1 \to \mathrm{SM}_{\mathrm{low}}$ is configuration-independent (constant); (B) the Cramér-Bernoulli LDP near the vacuum $\bar{X} = 1/2$ gives a Gaussian weight in $(\bar{X} - 1/2)$, not exponential in $\bar{X}$ (numerically verified $I(x) = x\ln(2x) + (1-x)\ln(2(1-x)) \approx 2(x-1/2)^2$ near $1/2$). The multiplicative matching $\lambda_{\mathrm{SM}}(M_*) = b \cdot Z_H(\beta_{\mathrm{KO}})$ is therefore NOT derivable from a path-integral change-of-variable. Confirms that the *Jacobian* route to (B) does not close. Sub-question (2) $[Z_{\Gamma 4} = 1]$ is satisfied at tree level (standard CC inner fluctuations). Net: this is a negative result for the measure-Jacobian reading specifically; it does *not* obstruct (B), because (B) is a *wave-function* renormalisation (multiplicative), not a coupling-matching Jacobian (additive). The Gaussian found here is precisely what the product-structure argument later excludes for the field-strength kernel (Comp 111), and (B) is subsequently closed within PST in the substrate limit by the wave-function route (Comps 110–116).

Comp.	Result
104 §7.23	**Bridge premise (B) factored: substrate side derived, SM side inherited from standard CC literature.** Attempts to extend Mosco convergence from Dirichlet forms (which sec:mosco-conditional establishes) to spectral actions, with the goal of deriving bridge premise (B) at the same level Mosco delivers the SM kinetic term. The structural analysis shows (B) FACTORS into two parts: (B-substrate) $(1/2^{\lvert D\rvert})\,\mathrm{Tr}\, f(\Delta/\lvert D\rvert) \to e^{-1}$, which is CLOSED at proof-detail level from P1–P3 by Comp 100; and (B-SM) the SM-side spectral-action identification of the matched-scaling coefficient with $\lambda_{\mathrm{SM}}(M_*)$, which IS the standard Chamseddine-Connes spectral-action principle [49, 50] applied to PST's substrate spectral triple. This is an ESTABLISHED FRAMEWORK with 30+ years of literature precedent, NOT a PST-specific postulate. PST extends standard CC by providing a precausal foundation BENEATH the spectral triple (P1–P3 deliver the substrate spectral triple itself, Comps 3, 19); it does not invent the SM-side framework. Whether the partition-function-level CC framework has standing in the spectral-action literature: YES, the framework PST uses IS standard CC, with the partition-function-level adaptation being PST's specific way of handling the discrete substrate where heat-kernel collapses (Comps 85–86). PST's foundational claim refines to "P1–P3 + standard CC spectral-action framework": the substrate-side contribution is the genuinely new PST result (Comp 100, derived from P1–P3), and the SM-side framework is inherited from established CC literature. *This inheritance framing is subsequently superseded*: the matching (B) is closed *internally* to PST, in the substrate limit, by the wave-function-renormalisation route and the unique-soft-mode theorem (Comps 110–116), without inheriting an SM-side CC postulate; the foundational claim is then "P1–P3 + matched scaling", the bridge being a derived consequence rather than an inherited framework.

Comp.	Result
105 §7.23	**Two-layer one-loop diagram (no double counting of M_* zero modes).** Addresses the concern of potential double-counting between the substrate $\chi_\emptyset$ singleton-cylinder order parameter and the emergent SM Higgs field C_H from Mosco convergence. The PST partition function factorises as $Z_{\text{PST}}(M_*) = Z_{\text{sub}}(M_*, \Lambda{=}\sqrt{D}) \cdot Z_{\text{EFT}}(M_*, \text{IR})$, with the two layers integrating over disjoint momentum ranges $[M_*^2, D]$ and $[m_h^2, M_*^2]$ that meet only at the single matching scale M_* (measure zero in both integrals). Three independent non-double-counting criteria are established: (i) disjoint momentum support; (ii) distinct dynamical variables ($\chi_\emptyset$ on $\mathcal{P}(D)$ vs C_H on $M = \mathbb{R} \times S^3$); (iii) distinct counterterm structures (the matching is a boundary condition, not a loop integration). Numerical verification: the substrate full-trace partition function (Comp 100) decomposes exactly as $Z_{\text{sub}}^{\text{full}}(D) = Z_{\text{sub}}^{\text{IR}}(D, k_*) + Z_{\text{sub}}^{\text{UV}}(D, k_*)$ across the M_*-shell partition with $\lvert Z^{\text{full}} - (Z^{\text{IR}} + Z^{\text{UV}})\rvert < 10^{-14}$ for $D \in \{10, 20, 50, 100\}$. Closes the double-counting question at the diagram level.
106 §8.19	**SM gauge block-diagonality on $\oplus_k G_k$.** Provides the explicit proof that SM gauge generators preserve each generation sector in the Dixon-synthesis decomposition $V_{\text{Dixon}} = V_{\text{core}} \oplus G_1 \oplus G_2 \oplus G_3$. Three-part proof: (1) $\text{SU}(3)_c$ is the $M_3(\mathbb{C})$ factor of A_F acting on the colour triplet WITHIN one Fano-plane sub-algebra H_k of $\mathbb{O}$ through $\hat{\tau}$, hence within a single G_k; (2) $\text{SU}(2)_L$ is the $\mathbb{H}$ factor acting on the quaternionic slot of $V_{\text{Dixon}} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, orthogonal to the octonionic generation labelling, hence block-diagonal by tensor-product structure; (3) $\text{U}(1)_Y$ is the $\mathbb{C}$-factor scalar action, same tensor-product argument. Numerical verification on a 12-dim toy model $V_{\text{gen}} \otimes V_{\text{per-gen}}$: $\text{SU}(3)_c$, $\text{SU}(2)_L$, $\text{U}(1)_Y$ all give machine-zero cross-generation matrix elements; a hypothetical "gauge on generation slot" control gives off-block magnitudes of $O(1)$. Concludes that the absence of tree-level flavour-changing neutral currents is a structural theorem of PST conditional on the Dixon-synthesis block structure, not a postulate.

Comp.	Result
107 §7.21	**Full-SM extension of the M_* boson-fermion cancellation: sector-blindness argument (now WITHDRAWN as a closure).** *Status (v26.15): the closure claim is withdrawn; open-research item 1.3 is reopened.* This computation established that the substrate-level binomial cancellation $N_B = N_F = 2^{D-1}$ is SECTOR-BLIND (invariant under any partition of the 2^D Walsh modes; numerically verified across $D \in \{4, \ldots, 20\}$) and argued that this transfers the cancellation to the SM top/W/Z sectors. The transfer does *not* hold: by the disjoint-layer structure of Comp 105, the substrate cancellation lives in the UV shell $[M_*, \sqrt{D}]$ while the physical top/W/Z loops run in the emergent EFT shell below M_*, so the substrate identity cannot cancel the EFT-layer loops. Moreover the "coupling universality at M_*" premise ($c_S = 1$ for every Walsh mode) is contradicted by the observed Yukawa hierarchy, which the paper itself assigns to $T(C)$-contingency (§8.20). The empirical SM Veltman residual at M_* is ≈ -3 (top-dominated, negative), not the $+m_h^2$ a clean cancellation would leave. The sector-blindness of the substrate count survives as a fact; its use to close the full-SM extension is retracted. The full-SM cancellation is an open problem.
108 §7.21	**Inner-fluctuation sign assignment for SM gauge bosons is BOSONIC (sub-result; does not close 1.3).** Verifies that the substrate $\|S\|$-parity boson/fermion sign assignment (CAR realisation, paper sec:car-fermions) carries through to A_F inner-fluctuation gauge bosons that are not themselves Walsh modes. Three independent readings agree: (R1) algebraic — Connes 1-forms $A = \sum_i a_i[D, b_i]$ for $a_i, b_i \in A_F$ are operators of EVEN Clifford grade; (R2) substrate-side lift — A_F is generated by $\mathrm{Cl}_{\mathrm{even}}(\mathcal{P}(D))$ under the CAR realisation, so every inner fluctuation lifts to an $\|S\|$-even substrate configuration; (R3) one-loop diagrammatic — the standard QFT vector-boson loop sign (+) is inherited unchanged by the CC inner-fluctuation construction. Dof count: bosonic inner-fluctuation dofs on A_F (off-shell) are $\mathrm{U}(1)_Y$: 4, $\mathrm{SU}(2)_L$: 12, $\mathrm{SU}(3)_c$: 32, Higgs doublet: 4, total 52; the substrate $\|S\|$-even Walsh subspace has dimension 2^{D-1} accommodating these at $D \geq 6$. *Status (v26.15):* the bosonic sign assignment is a correct sub-result, but it does NOT close open-research item 1.3. The full-SM cancellation fails for the layer-disjointness reason of Comp 107 (the substrate cancellation does not reach the EFT-layer top/W/Z loops), independent of the inner-fluctuation sign. Item 1.3 is reopened.

Comp.	Result
109 Chapter 7.21	**Higgs-quartic normalisation + O(4) one-loop self-energy: explicit verification of two referee findings.** Verifies two technical findings on the Higgs-sector derivation at proof-detail level. *Finding A (normalisation):* the paper commits to the *doublet convention* – $\lambda_{\mathrm{PST}} = \frac{1}{4}$ is the coefficient of $(\Phi^\dagger\Phi)^2$, the same convention as the SM $V_{\mathrm{SM}} = \lambda_{\mathrm{SM}}(\Phi^\dagger\Phi)^2$, with $\langle\psi,\psi\rangle \leftrightarrow \Phi^\dagger\Phi$. Under the radial reduction $\langle\psi,\psi\rangle \to \psi^2$ the postulate $\mathcal{F} = \int[-\varepsilon\langle\psi,\psi\rangle + (1/4)\langle\psi,\psi\rangle^2 + c|\nabla\psi|^2]\mathrm{d}\mu$ collapses to the scalar form $-\varepsilon\psi^2 + (1/4)\psi^4$ (eq. (23)). The field-normalisation ratio $Z^2 = \lambda_{\mathrm{SM}}(M_*)/\lambda_{\mathrm{PST}}$ is then a ratio of two same-convention couplings. *Finding B:* explicit O(N) vertex contraction $V_{ijkl} = 2\lambda(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ closed into the one-loop seagull $(1/2)V_{ijkl}\delta_{kl} = (N+2)\lambda\,\delta_{ij}$ delivers the standard O(N) result $(N+2)\lambda$ rather than $N\lambda$. For $N = 4$ the Higgs self-loop coefficient is $C_H = 6\lambda_{\mathrm{PST}} = 3/2$, giving $M_* = 4\pi m_h\sqrt{2/3} \approx 1285$ GeV. Broken-phase cross-check: 3λ (physical Higgs loop) $+3\lambda$ (three Goldstones) $= 6\lambda$ confirms the symmetric-phase result. This computation propagates corrections through eq. (107), the M_* value, the naturalness coefficient, and the affected Comps 67, 69, 93, 97, 107, 108.
110 §7.23	**Bridge premise (B): wave-function-renormalisation route (M1, M2).** Recasts (B) as a field-strength (wave-function) renormalisation Z_ϕ^2, which acts multiplicatively on the quartic ($\lambda \to \lambda Z_\phi^2$), the form (B) requires – sidestepping the additive-matching obstruction Comp 103 found for the coupling-Jacobian reading. M1 sets up the discrete Wilson–Polchinski flow on the Boolean lattice with the P1-forced cutoff $f = \exp$ and matched scaling $\Lambda^2 = D$, reducing to the low Walsh shells $V_0 \oplus V_1$ where $\bar{X}$ lives (Comp 98). M2 verifies that the field-strength factor is $Z_\phi^2 = Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$, so $\lambda_{\mathrm{SM}} = b\,Z_\phi^2 \to e^{-1}/4$. Sets up the route; the crux (the form and value of the kernel) is closed in Comps 111–116.
111 §7.23	**Bridge premise (B): the field-strength kernel is forced into $\exp(-\beta\bar{X})$ form (M3).** The field-strength dressing must respect P1's product measure (the matched-scaling cutoff factorises over bits, Comp 100), so the per-configuration kernel $K(C) = \prod_a g(C_a)$ is a product over bits. A product kernel over Boolean bits is necessarily exp of a quantity *linear* in $\bar{X}$, i.e. $\propto \exp(-\beta\bar{X})$ – the same form the substrate side evaluates to e^{-1}. The Gaussian $\exp(-\alpha(\bar{X} - 1/2)^2)$ that obstructs the Jacobian route (Comp 103) is thereby excluded: it correlates the bits, breaking the product structure (verified: bit–bit covariance 0 under $\exp(-2\bar{X})$, nonzero under the Gaussian). The wave-function route is therefore unobstructed, in contrast to the Jacobian route.

Comp.	Result
112 §7.23	**Bridge premise (B): $\beta_{\mathrm{KO}} = 2$ is the binary-distinction gap (M4a).** The load-bearing exponent is the spectral range of a single binary distinction, $2 = 1-(-1)$: each exchange τ_i is a self-adjoint involution with spectrum $\{+1, -1\}$, so $(1-\tau_i)$ has nonzero eigenvalue 2, forced by P1's binary alphabet with no freedom (equivalently the one-bit Clifford $\mathrm{Cl}(1,0)$ range). q-ary comparison: only $q = 2$ gives the self-adjoint involution P1 requires, and e^{-1} is the signature of the binary gap ($Z_H \to e^{-\beta/2}$ lands on e^{-1} only at $\beta = 2$). Dissolves the "two 2's" question: $\mathrm{KO}_{\mathrm{total}}$ mod $8 = 2$ shares the value but is a gauge/spacetime-sector fact, never an input.
113 §7.23	**Bridge premise (B): the $Z_\phi \leftrightarrow$ trace identification, total-reduction residual (M4b).** Tests the field-strength↔matched-scaling-trace identification ab initio. The order-parameter two-point residue (single $\lvert S\rvert = 1$ shell) gives a single-mode factor, not the full e^{-1}, unless all substrate modes are folded in. The matched-scaling trace and the tempered order-parameter expectation $\mathbb{E}_\mu[\exp(-2\bar{X})]$ are verified exactly equal, so the identification holds iff the substrate→Higgs reduction is *total*. Isolates the residual as one concrete property (total reduction), closed by Comps 114–116.
114 §7.23	**Bridge premise (B): the standard-EFT normalisation is excluded; embedding-norm reading (M5).** Two field-theoretic normalisations of the reduction: (A) the total-reduction embedding norm of the symmetric Higgs vacuum $= \mathbb{E}_\mu[\exp(-\beta_{\mathrm{KO}}\bar{X})] \to e^{-1}$, verified exactly equal to the tempered vacuum amplitude; (B) the standard-EFT two-point residue $\mathrm{Var}_\mu(\bar{X}) = 1/(4D) \to 0$. The bridge needs (A); route (B), the Higgs as a light field whose two-point function is corrected by heavy loops, gives $\lambda_{\mathrm{SM}} \to 0$ and is excluded by the finiteness of the observed quartic. The Higgs is an emergent projected field, normalised by its embedding norm, not a fundamental light field.
115 §7.23	**Bridge premise (B): unique-soft-mode theorem, total reduction (M6).** Past the LG threshold (P3) the order parameter $\bar{X}$ is the unique soft mode. Lemma: $\bar{X} = (1/D)\sum_a C_a$ is linear in the bits, so the LG Hessian is rank one (the all-ones matrix), softening exactly the S_D-singlet direction ($= \bar{X}$) and leaving the $D-1$ standard-rep modes and all $\lvert S\rvert \geq 2$ shells at the binary gap. Verified: rank $= 1$, one soft eigenvalue, $D-1$ pinned at the gap; the threshold sweep softens $\bar{X}$ alone. Establishes total reduction (Theorem F11, Gaussian order), so the bridge factor is the full embedding norm e^{-1}.

Comp.	Result
116 §7.23	**Bridge premise (B): all-orders decoupling theorem (M7).** Upgrades the unique-soft-mode result to all orders. Because the LG potential is a function of the linear collective coordinate $\bar{X}$ alone, every derivative tensor of V is the totally symmetric all-ones tensor $V^{(n)}/D^n$, with zero matrix element on the non-collective subspace at every order: the standard-rep and $\lvert S\rvert \geq 2$ modes are free fields, decoupled, pinned at the binary gap, with no V-vertex from which any correction could be built. Verified exactly with the full $\exp(-V)$: the V-induced bit correlation is $O(1/D)$, the standard channel sits at the free binary-gap value with $O(1/D)$ deviation, and only the singlet $\bar{X}$ responds to V. The spectral gap is protected non-perturbatively in the substrate limit; the residual is the standard $O(1/D)$ finite-size correction.

Appendix D: Claims Status

This appendix consolidates every substantive claim of the paper into a single structured inventory. Per-section "Claim status of § N" blocks at the end of each section remain the authoritative source of detail; this appendix is the cross-cutting index. Each item is tagged with one of seven status categories defined below.

Status categories.

Postulate. An ontological input; PST has exactly three (P1, P2, P3).

Modelling choice. An item adopted within the framework where alternatives exist; documented and not derived from postulates alone.

Derived theorem. Proved from P1–P3 with all conditions in place at proof-detail level.

Theorem under modelling choice. Proved given a modelling choice; result depends on that choice.

Conditional theorem. Proved under explicit assumptions (e.g. A1–A6 of §7.7).

Identification. Structural correspondence between substrate objects and Standard-Model / GR / QFT counterparts.

T(C)-contingent. Configuration content; NOT derivable from postulates alone (structural-scope theorem of §8.20).

D.1 Postulates

Postulate	Reference	Statement
P1 – Property differentiation	§1.5	The capacity for distinctions to exist; sole ontological primitive.
P2 – Asymmetric vector tension	§3	Property differentiation carries a V_7-valued tension with $T(D) \neq 0$ generically.
P3 – Modal sublimation	§4.1	G_2-invariant Landau–Ginzburg potential on V_7; instantiated geometry follows past threshold.

D.2 Modelling choices

Choice	Reference	Alternatives & rationale
$V_7 = \mathrm{Im}(\mathbb{O})$ as direction algebra	§1.5	Maximal directional structure; proper sub-algebras (e.g. V_3) not considered.
Gaussian projection kernel for Casimir ξ	Chapter 10, Comp 68	Six kernel families surveyed; Gaussian gives $\xi = 90/\pi^2$; substrate's binomial kernel converges to Gaussian (Comp 73).

D.3 Derived structural theorems (from P1–P3 at proof-detail level)

Theorem	Reference	Notes
Bernoulli measure $\mu = \bigotimes_a \mathrm{Bern}(\frac{1}{2})$ as unique permutation-invariant measure on $\mathcal{P}(D)$	§1.5	Derived from P1's distinguishability + symmetry.
Modal threshold τ exists with specified value	§4.1	Bifurcation point of $F[\psi, \varepsilon]$ from P2 + P3.
Vacuum manifold $\mathcal{V} \cong S^6 \subset V_7$ for $\varepsilon > 0$	§4.1	G_2-orbit of vacuum direction.
Lorentzian signature $(+,-,-,-)$	Comp 2, §4.16	From hyperbolic well-posedness of Goldstone wave equation.
Four-dimensional spacetime $n = 4$	§7.10	From minimal Connes-spectral-triple split given KO-6 substrate.
Diffeomorphism invariance	§4.16	From non-uniqueness of realisation map ρ.
Geroch time-orientability of emergent manifold	§3	P3-directed threshold + permutation invariance + Mosco convergence.
Cosmological principle	§11.5	From permutation invariance of μ.
SM gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	Chapter 8, Comp 6	Unimodular unitaries of A_F.
Internal algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$	§8.8, Comps 3, 19	From Cl(0,6) chain-algebra structure.

Theorem	Reference	Notes
Three-generation count $N_{\text{gen}} = 3$ (structural reading)	§8.19	V_7-directional decomposition into three quaternionic sub-algebras through $\hat{\tau}$; the natural structural reading, conditional on the Dixon-synthesis block structure (Comp 48, 106) and subject to the colour–generation distinguishability caveat of the octonionic-SM programme (Furey 2014).
Born rule	§9.2	From Gleason's theorem on $L^2(\mathcal{P}(D), \mu)$.
Fermionic excitations	§8.15	Boolean-CAR isomorphism on distinction primitive.
Spin-statistics structural	§8.16	Three closures: CAR Mosco convergence, Wightman verification, substrate-derived Dirac equation.
Collapse rule (measurement postulate)	§9.3	Three closures: von Neumann measurement Hamiltonian, decoherence inheritance, threshold-crossing irreversibility.
QCD vacuum angle $\theta_{\text{QCD}} = 0$ structurally	Chapter 8	From substrate's parity structure.
Substrate-side factor $Z_H(\beta_{\text{KO}}) \to e^{-1}$ (substrate side of $Z^2 \approx e^{-1}$)	§7.23, Comp 100	Tensor-product factorisation forced by P1's Bernoulli product measure (substrate-side spectral-action cutoff identification). The matching identification with the SM-side ratio $\lambda_{\text{SM}}(M_*)/b$ – bridge premise (B) – is closed within PST in the substrate limit (§7.23, Comps 110–116).
Cosmological-constant sign + equation of state $w = -1$	§11.5	Steady-state non-dilution mechanism.
d^{-6} Casimir-correction power law	Chapter 10	Categorical theorem from parity + single scale d_0.

D.4 Theorems under modelling choice

Theorem	Reference	Modelling choice
$\xi = 90/\pi^2 \approx 9.12$ Casimir coefficient	Chapter 10, Comp 73	Gaussian projection kernel (substrate kernel converges to Gaussian at matched scaling, correction $-2/(3\|D\|)$).
$d_0 \lesssim 5.3$ nm coherence-length bound	Chapter 10	Conditional on Gaussian kernel; factor-of-$\sim$3 spread across kernel families.

D.5 Conditional theorems (explicit assumptions listed)

Theorem	Reference	Conditions
Mosco convergence $\mathcal{S} \to M = \mathbb{R} \times S^3$	§7.7	Six assumptions A1–A6. A6 (equicoercivity) reduced to uniform-rate hypothesis (R) in §7.7 with three closure subsections delivering (R) at proof-detail level (Comps 70, 83).
Stress-energy = projected tension; EFE as Seeley-DeWitt term	§7.10, Comp 5	Standard CC spectral-action machinery applied to substrate triple.
Newton's G as consistency relation among (m_h, M_P, v)	§7.10	Conditional on heat-kernel CC matching.
$m_h^2 = (3/2)M_*^2/(16\pi^2)$ scalar self-mass; $M_* = 4\pi m_h\sqrt{2/3} \approx 1.29$ TeV	§7.21, Comps 93, 97, 109	Parameter-free one-loop result *for the Higgs self-loop in isolation*. Conditional on the EFT-layer top/W/Z quadratic sensitivities being separately accounted for, which is *not* established (open item 1.3) and is in tension with the large negative Veltman residual eq. (104). The earlier "full-SM extension closed" claim (Comps 107, 108) is withdrawn.
$Z^2 \approx e^{-1}$ field-normalisation identity / Bridge premise (B)	§7.23, Comps 100, 110–116	Substrate-side factor $Z_H(\beta_{\mathrm{KO}}) \to e^{-1}$ derived from P1–P3 (D.3, Comp 100). Matching identification with $\lambda_{\mathrm{SM}}(M_*)/b$ (bridge premise B) is CLOSED in the substrate limit: a wave-function (field-strength) renormalisation, multiplicative rather than the additive Wilsonian matching the measure-Jacobian route excludes (Comp 103), forced by P1's product structure into the substrate-factor form (Comp 111), equal to the embedding norm of the symmetric Higgs vacuum (Comp 114), with the total reduction supplied by a unique-soft-mode theorem (Comps 115, 116). Residual: an all-orders equivariant write-up of the decoupling lemma; standard $O(1/D)$ substrate-limit caveat. Empirical content: few-percent match (Comp 91).

D.6 Identifications

Identification	Reference	Content
Substrate modal field = SM Higgs at different scale	§10.4	$\phi_{\text{Higgs}} = \Pi(\psi)$, $m_h^2 = 4\varepsilon$.
Substrate Higgs Hamiltonian $H_{\text{Higgs}} = \bar{X}$	Comp 89, §7.23	From additivity + matched scaling + uniformity; no free parameters.
Matched-scaling exponent $\beta_{\text{KO}} = 2$	Comp 100, 102	Spectral range of a single binary distinction, $2 = 1 - (-1)$, forced by P1's binary alphabet (equivalently the one-bit Clifford $\text{Cl}(1,0)$ range). The KO Bott-periodicity value $\text{KO}_{\text{total}} \bmod 8 = 2$ shares the value but is not the load-bearing source.
Matched scaling $\Lambda^2 = D$	Comp 100	Substrate dimension as natural matched scale for Boolean Laplacian's spectrum.

D.7 T(C)-contingent (configuration content, not derivable from postulates alone)

Item	Reference	Status
Yukawa mass hierarchy (quarks + charged leptons)	§8.20, Comps 76–80	T(C)-contingent; structural-scope theorem (Comp 1) forces S_3 symmetry at leading order.
CKM mixing matrix	§8.20	T(C)-contingent; same structural-scope theorem.
Neutrino masses + PMNS matrix	§8.20	T(C)-contingent.
d_0 coherence-length magnitude	§13.11	LG-dynamics quantity, not pinned by P1–P3 alone.
Λ_{obs} cosmological-constant magnitude	§13.14, Comp 82	T(C)-contingent. Sign and equation of state $w = -1$ remain structural theorems.

Appendix E: Comparison Other Substrate & Emergent Physics Theories

This appendix tabulates the substrate-style and emergent-physics theories that share posture with PST: each posits a layer beneath spacetime, causality, or the Standard Model from which the macroscopic structure is supposed to follow. The narrative comparison sits in Chapter 12; the table below condenses each theory along five axes for side-by-side reading.

Inclusion criterion. A theory is included if it either (i) posits a pre-geometric primitive from which spacetime or causality is derived, or (ii) derives a Standard-Model-relevant structure from an underlying programme. Theories PST recovers as effective descriptions (General Relativity, Quantum Mechanics, Quantum Field Theory) are not in the comparison set; for those see Chapter 12.

Axes. The columns are: the substrate primitive (what is taken as fundamental); the spacetime-emergence mechanism (how, if at all, spacetime is derived); Standard-Model coverage (whether gauge group, matter content, or families are addressed); and the relation to PST (whether PST agrees, extends, contrasts, or sits orthogonal). Entries are deliberately compressed; the cited references give the full treatment.

Theory	**Substrate primitive**	**Spacetime emergence**	**Standard-Model coverage**	**Relation to PST**
Reference row.				
PST (Meelker, 2026)	Property differentiation with V_7-valued directional content (D, δ, v)	Mosco convergence of V_7-Boolean Dirichlet form to Laplace–Beltrami on $\mathbb{R} \times S^3$ along $\hat{\tau}$	$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ derived; one generation from $\mathrm{Cl}(0,6)$ minimal ideal; $N_{\mathrm{gen}} = 3$ from quaternionic sub-algebras of $\mathbb{O}$ containing $\hat{\tau}$	—
Pre-geometric / combinatorial substrate.				

Theory	Substrate primitive	Spacetime emergence	Standard-Model coverage	Relation to PST
Wolfram Physics Project (Wolfram, Gorard, 2020+) [122]	Spatial hypergraph with local rewrite rules	Causal graph from update sequence; (3+1)D from rule-space exploration	Programmatic; gauge groups not yet derived from rules	Closest in posture (no presupposed geometry; everything from substrate updates); differs on substrate nature (computation vs. distinction-plus-direction); PST derives SM from internal algebra rather than searching rule space
't Hooft CA Interpretation ('t Hooft, 2014) [123]	Deterministic Planck-scale cellular automaton	Spacetime lattice presupposed; QM emerges from CA bookkeeping	Programmatic; SM not derived	Shares "QM is derived" thesis; PST's Mosco-convergence route delivers CCR from substrate without requiring determinism
Group Field Theory (Oriti, Freidel, Pereira, 2000+) [124]	Second-quantised spin-network vertices on a Lie group	Spacetime as condensate phase of GFT (Bose–Einstein-like)	Silent on matter content	Combinatorial substrate analogue of PST's algebraic substrate; both invoke limit theorems; GFT yields LQG-style quantum geometry, PST yields a Connes spectral triple
Causal Set Theory (Bombelli–Lee–Meyer–Sorkin, 1987+) [4, 5]	Locally finite partially ordered set of events	Spacetime continuum as dense Lorentz-invariant "sprinkling" limit	Silent on matter	Pre-geometric combinatorial substrate; PST adds V_7-valued directional content beyond bare order, supplying gauge structure absent in causal sets

Theory	Substrate primitive	Spacetime emergence	Standard-Model coverage	Relation to PST
Loop Quantum Gravity (Rovelli, Smolin, 1986+) [6]	Spin-network states quantising SU(2) holonomies and fluxes	Continuum spacetime as semiclassical limit of spin-foam histories	Not derived; matter coupled by hand	Spin-network combinatorial substrate; PST's substrate is algebraic (Walsh modes / Cl(0,6)) rather than spin-network; both aim at background-independent quantum geometry
Penrose Twistor Theory (Penrose, 1967+) [125]	Twistor space $\mathbb{T} \cong \mathbb{C}^4$ encoding null geodesics + helicity	Spacetime points as projective null lines in twistor space (Klein correspondence)	Conformal kinematics; gauge structure later via twistor-string/amplituhedron	Pre-spacetime spinor programme; PST's distinction substrate is logically prior to both spacetime and the twistor encoding
Energetic Causal Sets (Cortês–Smolin, 2014) [126]	Events with energy–momentum + causal poset	Spacetime as derived from energetic causal poset	Programmatic	Extends causal sets with dynamical input; PST has richer substrate (V_7 directionality + Bernoulli measure) and a spectral-triple emergence route
Hardy Causaloids (Hardy, 2005+) [127]	Causaloid: operational data structure encoding probabilistic causal correlations	Causal structure primitive; geometry derived	Operational, silent on specific gauge/matter	Operational pre-geometric framework; PST is an ontological pre-geometric framework with explicit derivation chain
Causal Fermion Systems (Finster, 2010+) [128]	Measure on operators of finite spatial rank in a Hilbert space (no a priori spacetime)	Spacetime + causal structure emerge from operator-support overlap; Lorentzian signature from causal action principle	Derives Dirac sea + a Standard-Model-like gauge structure (chiral fermions, weak isospin) from the causal action	Closest mathematical kin: both use Hilbert-space-with-extra-structure as substrate; CFS uses measure theory + causal action, PST uses Connes spectral triples + Mosco convergence; CFS gauge derivation runs through a different mechanism

Theory	Substrate primitive	Spacetime emergence	Standard-Model coverage	Relation to PST
BFSS Matrix Models (Banks–Fischler–Shenker–Susskind, 1996) [129]	Large-N matrices of $D0$-brane positions and spinor partners	11-D supergravity as the $N \to \infty$, 't Hooft limit; spacetime as eigenvalue distribution	Low-energy SUGRA spectrum	Matrix-model substrate; PST is spectral-triple substrate; both invoke large-N/coarse-graining limits but in mathematically distinct categories
General Substrate Theory (GST)	Causal pacing in place of geometry; physical laws as coherence-conserving structural adaptations	Spacetime replaced by causal-pacing relations; relativistic and quantum-discreteness phenomena recovered from coherence-conservation constraints	Programmatic; not specific on gauge/matter content	Shares pre-geometric posture; PST commits to a specific substrate (V_7-directional distinguishability) and a definite emergence theorem (Mosco convergence) rather than an adaptive coherence-pacing principle
Pre-geometric Substrate Theory (RPST)	"Linkons": primitive relations that self-organise	Continuous information fields and gravity emerge from coherence among linkons rather than from a pre-existing spatial manifold	Programmatic	Closest in stated spirit – "relations as primitive" parallels PST's distinguishability; PST commits to specific algebraic structure (V_7, Bernoulli measure, Landau–Ginzburg potential) and a Connes spectral-triple realisation, where RPST as described remains schematic

Theory	Substrate primitive	Spacetime emergence	Standard-Model coverage	Relation to PST
Substrate Theory of Everything (Reed, 2024) [141]	Single scalar substrate field with Mexican-hat potential and U(1) gauge structure; eight ontological postulates; two-frame ontology distinguishing undisturbed substrate (F_1) from perturbed observable form (F_2)	Spacetime emerges from substrate perturbations; Newton, special relativity, general relativity recovered as theorems	Standard-Model gauge structure and particle-mass cascade derived from U(1) cascade; dark matter / dark energy and matter–antimatter asymmetry addressed; testable predictions at HL-LHC, Gaia DR4, CMB B-modes, LISA	Most parallel in posture: another substrate from which spacetime, gravity, and SM physics are derived, with a Mexican-hat-style potential at the centre. Differences are technical: Reed uses a scalar substrate with a single U(1) producing the SM gauge cascade, where PST uses a V_7-directional distinguishability substrate, G_2-invariant LG potential on V_7, and the Connes spectral-triple route; PST's gauge group emerges as the unimodular unitaries of $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (itself derived from P1–P3) rather than from a single U(1) cascade
Emergent-gravity programme: gravity as effective description of underlying microphysics.				
Sakharov Induced Gravity (Sakharov, 1967) [130]	Standard-Model quantum-field vacuum fluctuations	Einstein–Hilbert action as one-loop effective action of matter fields	Presupposes SM matter	Historical precursor of all emergent-gravity proposals; PST replaces "vacuum fluctuations" with a precausal substrate and derives the SM rather than presupposing it

Theory	Substrate primitive	Spacetime emergence	Standard-Model coverage	Relation to PST
Thermodynamics of Spacetime (Jacobson, 1995) [19]	Local Rindler-horizon degrees of freedom (unspecified)	Einstein equation derived as the Clausius equation of state $\delta Q = T\mathrm{d}S$ on local horizons	Not addressed	Parallel "gravity from underlying microphysics" claim; PST specifies the microphysics (V_7-substrate) and recovers the same Einstein equation via the spectral action
Atoms of Spacetime (Padmanabhan, 2002+) [131]	Microscopic "atomic" degrees of freedom of spacetime horizons	Gravity as horizon thermodynamics; equipartition over the surface degrees of freedom yields Einstein equations	Not addressed	Same emergent-gravity claim as Jacobson; PST's route is spectral-action (matter + gravity coupled at leading order) rather than horizon-thermodynamic
Entropic / Emergent Gravity (Verlinde, 2010, 2016) [18, 132]	Holographic information / quantum entanglement between regions	Gravity as entropic force from holographic counting; modified inertia in low-acceleration regime	Dark-matter phenomenology suggested; SM not derived	Emergent-gravity programme; PST's gravity emerges from spectral action on the substrate, not from entanglement entropy bookkeeping
Quantum-information substrate: spacetime from entanglement.				
ER=EPR / It from Qubit (Maldacena–Susskind 2013; Van Raamsdonk 2010) [133, 134]	Entanglement structure of the boundary quantum theory	Bulk geometry from entanglement entropy (Ryu–Takayanagi); ER bridges as EPR pairs	Inherits boundary CFT's matter content; not derived	Spacetime-from-entanglement vs. PST's spacetime-from-Boolean-Dirichlet-Mosco; complementary programmes; PST gives a substrate finite in internal structure and an explicit limit theorem, IfQ relies on holography

Theory	Substrate primitive	Spacetime emergence	Standard-Model coverage	Relation to PST
Tensor-Network Emergence (Swingle 2009; Vidal MERA) [135, 136]	Hierarchy of tensor-network entanglement (MERA / HaPPY codes)	AdS geometry from MERA scale layers; bulk locality from network connectivity	Silent on matter content	Operational realisation of It-from-Qubit; geometry from QI structure rather than from a logical primitive
Algorithmic Information Substrate	Algorithmic information / computational complexity of the substrate's update history	Quantum vs. classical regime separated by a critical algorithmic-complexity threshold ($\sim$ 50 bits in some proposals): below threshold the substrate maintains quantum superposition; above it, classical irreversibility	Not derived; treated as effective consequence of complexity-threshold crossing	Complementary in mechanism: PST yields the quantum/classical transition via Mosco convergence of Boolean Dirichlet forms and the modal threshold τ rather than via an algorithmic-complexity cut-off; both share a discreteness $\to$ continuum scaling structure
Foundational / philosophical substrate proposals.				
Octonion-Substrate Programme (Dixon 1994; Manogue–Dray 1999+; Furey 2014+) [137, 138, 120, 121]	Division-algebra chain (e.g. $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$); octonions as fundamental	Spacetime not derived; programme is internal-algebra-focused	One generation of fermions from minimal ideal of Cl(6); SU(3) $\times$ SU(2) $\times$ U(1) from unimodular unitaries	Closest *technical* kin: PST extends Furey 2014 by deriving $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ from P1–P3 (rather than positing the chain algebra) and pairs the internal algebra with explicit spacetime emergence; PST further derives $N_{\text{gen}} = 3$ from quaternionic subalgebras of $\mathbb{O}$ containing the modal-sublimation direction $\hat{\tau}$, an item the octonion programme has not closed

Theory	Substrate primitive	Spacetime emergence	Standard-Model coverage	Relation to PST
Constructor Theory (Deutsch & Marletto, 2013+) [139]	Counterfactual possible/impossible transformations	Spacetime presupposed; not derived	Not addressed	Shares "logical primitive" stance with PST's distinguishability postulate; orthogonal mathematical machinery (possibility/impossibility lattices vs. Connes spectral triples)
Spencer-Brown Distinction Ontologies (Spencer-Brown, 1969; Kauffman; Varela) [61]	The act of making a distinction	Not formalised; mostly philosophical	Silent	PST is the closest *formalisation* of "distinction as primitive" with an explicit derivation chain to physics; shares the foundational starting point but commits to additional structure (V_7 directionality, Bernoulli measure, Landau–Ginzburg potential)
Wheeler's "It from Bit" / MUH (Wheeler 1989; Tegmark 1998+) [62, 63]	Binary informational acts (Wheeler) / all mathematical structures (Tegmark)	Programmatic; no specific derivation	Programmatic	Programmatic ancestor; PST instantiates the philosophy with concrete substrate + derivation chain rather than leaving emergence as a slogan

Theory	Substrate primitive	Spacetime emergence	Standard-Model coverage	Relation to PST
Information Substrate Theory (Kriger, 2026) [140]	"Information" as the name for *differentiatedness* in being; a single axis-free substrate of which Shannon entropy, Kolmogorov complexity, thermodynamic entropy etc. are specific projections (along time, space, energy, ...)	Spacetime not derived; affirmed as one projection / coordinate reading of the same substrate	Affirms existing SM; not derived	Closest in *ontology*: both PST and IST take differentiatedness as the foundational primitive (PST's "property differentiation" ↔ IST's "information as the name of differentiatedness"). PST commits to a specific algebraic realisation (V_7-directional distinctions, Bernoulli measure, G_2-invariant Landau–Ginzburg potential, Connes spectral triple) and an explicit derivation of spacetime and the Standard Model; IST keeps the substrate axis-free and reads existing physics off it as projections rather than deriving the projections from postulates

Conclusion of comparison. PST is the only entry in the table that simultaneously (i) posits a finite pre-geometric substrate, (ii) derives spacetime as a limit theorem from it, (iii) derives the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge group from the three postulates, (iv) places $N_{\mathrm{gen}} = 3$ structurally, and (v) delivers a specific Standard-Model coupling value $\lambda_{\mathrm{SM}}(M_*) = b \cdot \mathrm{e}^{-1} = 0.0920$ matching the observed value at 0.8%. Other entries deliver one or two of these; none delivers all five.